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10. Vertex operators and the complex plane (19 points)

To be discussed on Thursday, 5th January, 2023 in the tutorial.
Please indicate your preferences until Saturday, 31/12/2022, 21:00:00 on the website.

Exercise 10.1: Vertex operators, OPE's and correlation functions

In the lecture you learned how one efficiently computes OPEs. Here, you can apply this knowledge to get more familiar with Vertex operators.

a) (2 points) Compute

$$:e^{ik_\mu X^\mu}:(z,\bar{z})X(0)\,.$$

b) (2 points) Compute

$$:e^{ik_\mu X^\mu}:(z,\bar{z})\partial X^\nu(0)\,.$$

Hint: This OPE can be computed in two ways: either (i) directly or (ii) by using a.

c) (2 points) Show that

$$:e^{ik_\mu X^\mu}:(z,\bar{z})$$

is a primary field with conformal wieghts $(\alpha' k^2/4\,,\ \alpha' k^2/4)$.

Hint: Hint: Calculate its OPE with the energy momentum tensor T.

d) (2 points) Compute

$$:e^{ik_\mu X^\mu}:(z,\bar{z}):e^{ik_\nu X^\nu}:(0)\,.$$

e) (3 points) Compute

$$\big\langle :e^{ik_1\cdot X}:(z_1,\bar{z}_1):e^{ik_2\cdot X}:(z_2,\bar{z}_2):e^{ik_3\cdot X}:(z_3,\bar{z}_3)\big\rangle\,,$$

which appears in the closed string tree amplitude for 3 tachyons.

Exercise 10.2: Asymptotic in and out states

Remember that a (chiral) primary field $\phi(z)$ with conformal weight h in the complex plane has the mode expansion

$$\phi(z)=\sum_{n\in\mathbb{Z}} z^{-n-h}\phi_n\,. \tag{1}$$

As $z\to 0$ corresponds to $\tau\to-\infty$ on the cylinder, a state of the form

$$|\phi_{\text{in}}\rangle := \lim_{z\to 0}\phi(z)|0\rangle$$

can be considered as asymptotic “in-state”. For this state to be non-singular, we need to impose

$$\phi_n|0\rangle = 0 \quad \forall n\geq 1-h\,, \tag{2}$$

as discussed in the lecture.

a) (1 point) Verify that, if (1) is satisfied, we have

$$|\phi_{\text{in}}\rangle = \phi_{-h}|0\rangle$$

with

$$\phi_{-h} = \oint_{C_0} \frac{dz}{2\pi i} \frac{\phi(z)}{z}$$

being the coefficients ϕ_n for $n = -h$.

b) (2 points) Using

$$[L_n, \phi_m] = [n(h-1) - m]\phi_{n+m}$$

and

$$L_n|0\rangle = 0 \quad \forall n \geq -1\,,$$

show

$$\begin{aligned} L_0|\phi_{\text{in}}\rangle &= h|\phi_{\text{in}}\rangle \\ L_n|\phi_{\text{in}}\rangle &= 0\,, \quad n > 0 \end{aligned}$$

holds. Therefore, $|\phi_{\text{in}}\rangle$ is a highest weight state of a representation of the Virasoro algebra called "Verma module".

Remark: The primary operators of a CFT are in one-to-one correspondence with highest weight states of Verma modules. A full Verma module is generated by acting with $L_{-n}(n > 0)$ on a highest weight state. The states so-generated are called descendant states. They can be generated directly from the vacuum by acting with descendant fields, which are operators that occur in (possibly multiple) operator products of the primary field with the energy momentum tensor.

c) (1 point) For a non-chiral primary field $\phi(z, \bar{z})$ with conformal weights $(h, \bar{h})$, one defines in a similar way

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle = \phi(0,0)|0\rangle\,,$$

which additionally results in

$$\begin{aligned} \overset{(-)}{L}_0|\phi_{\text{in}}\rangle &= \overset{(-)}{h}|\phi\rangle \\ \overset{(-)}{L}_n|\phi_{\text{in}}\rangle &= 0 \quad \forall n > 0\,. \end{aligned}$$

From this, read off what extra condition has to be imposed on the primary operator $\phi(z, \bar{z})$ for the state $|\phi_{\text{in}}\rangle$ to be physical in the sense of the old covariant quantization of the close string. (I.e., we assume that $\phi(z, \bar{z})$ is a primary operator in the conformal field theory that describes the quantized closed bosonic string).

Remark: Primary operators that satisfy the additional constraint to be found here are called vertex operators. They can be used to create physical in and out states from the vacuum. In string theory, scattering amplitudes of asymptotic in out out states are thus determined by vacuum expectation values of products of the vertex operators that correspond to those in and out states. The vertex operator for the close string tachyon, for example, is given by $:e^{ik_\mu X^\mu(z,\bar{z})}:$ with $k^2 = 2$, whereas the graviton, Kalb-Ramond field and dilaton corresponds to particular linear combinations of the vertex operators $:\partial X^\mu(z)\bar{\partial}\bar{X}^\nu(\bar{z})e^{ik_\rho X^\rho(z,\bar{z})}:$ with $k^2 = 0$. To verify that these are primary operators and that they indeed satisfy the additional requirement found here, one has to calculate the operator product with the (normal-ordered) energy momentum tensor using Wick's theorem for products of normal-ordered operators (see the lecture for more examples).

Exercise 10.3: The complex plane and the cylinder

A special case of a conformal/holomorphic transformation is the map

$$z' \to z = e^{z'}\,, \quad \bar{z}' \to \bar{z} = e^{\bar{z}'}\,,$$

which maps the cylinder (the Wick-rotated world sheet of a non-interacting closed string) to the complex plane.

a) (1 point) Calculate the rescaling function $f(z, \bar{z})$ introduced in 9.1 c) of the last exercise for this conformal transformation.

b) (1 point) What is the image of a curve of constant τ under this transformation?

c) (2 points) Determine how σ transformations $\sigma \to \sigma + \theta$ and time translations $\tau \to \tau + a$ operate on the new coordinates z and $\bar{z}$ on the complex plane and interpret the result geometrically.