

12.3. Dynkin index of embedding

Take $i: \mathfrak{h} \rightarrow \mathfrak{g}$ as the map which captures the embedding of $\mathfrak{h} \subset \mathfrak{g}$ in \mathfrak{g} .

Compute: Killing form $K_{\mathfrak{h} \subset \mathfrak{g}}(x, y) := K_{\mathfrak{g}}(i(x), i(y))$
 $x, y \in \mathfrak{h}$

and comparing with $K_{\mathfrak{h}}(x, y)$ gives:

$$c I_{\mathfrak{h} \subset \mathfrak{g}} := \frac{K_{\mathfrak{g}}(i(x), i(y))}{K_{\mathfrak{h}}(x, y)} \quad \text{with} \quad c = \frac{I_{\text{ad}}(\mathfrak{g})}{I_{\text{ad}}(\mathfrak{h})}$$

remember: $I_{\text{ad}}(\mathfrak{g}) / I_{\text{ad}}(\mathfrak{h})$ are the normalisations we introduced,

$$K_{\mathfrak{g}}(x, y) = \frac{1}{I_{\text{ad}}(\mathfrak{g})} \text{tr}_{\text{ad}}(\text{ad}_x \circ \text{ad}_y),$$

in section 4.6.

note: normalisation of Killing form fixed by highest weight.

$$\leadsto \boxed{I_{\mathfrak{h} \subset \mathfrak{g}} = \frac{(\Theta(\mathfrak{g}), \Theta(\mathfrak{g}))}{(\Theta(\mathfrak{h}), \Theta(\mathfrak{h}))} \begin{array}{l} \leftarrow \text{highest weight in } \mathfrak{g} \\ \leftarrow \text{and in } \mathfrak{h} \end{array}}$$

= Dynkin index of the embedding

For regular embedding: $\Theta(\mathfrak{h}) \in \text{roots of } \mathfrak{g}$

- \leadsto for simply laced $I_{\mathfrak{h} \subset \mathfrak{g}} = 1$
- \circ for B_n, C_n or F_4 $I_{\mathfrak{h} \subset \mathfrak{g}} \in \{1, 2\}$
- \circ for G_2 $I_{\mathfrak{h} \subset \mathfrak{g}} \in \{1, 2, 3\}$.

12.4. Special subalgebras

remember: $V = n$ dim module of \mathfrak{g}

$$\longrightarrow S_V: \mathfrak{g} \rightarrow \mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{C})$$

even more:

- 1.) \mathfrak{g} is simple $\rho_V(x) \in \mathfrak{sl}(n)$ and
- 2.) V is self-conjugated $\rho_V(x) \in \underbrace{\mathfrak{so}(n)}_{\text{for the adjoint}} \leftarrow \text{or } \mathfrak{sp}(n)$

for example 1.) $\rightarrow E_6 \hookrightarrow \mathfrak{sl}(27, \mathbb{C})$ and
2.) $\rightarrow \mathfrak{g} \hookrightarrow \mathfrak{so}(\dim \mathfrak{g})$
are maximal special embeddings

Semi simple embeddings

remember: $A_{n-1} \cong \mathfrak{su}(n)$ and the reducible module

$$V = V_{\square} \oplus V_{\bar{\square}} = \left. \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} n-1 \text{ with}$$
$$\rho_V \in \mathfrak{so}(\dim V) = \mathfrak{so}(2n)$$

$\rightarrow \mathfrak{sl}(n) \hookrightarrow \mathfrak{so}(2n)$ but not maximal
 \swarrow
 real version of $\mathfrak{su}(n)$

The corresponding maximal embedding is

$$U(1) \oplus \mathfrak{sl}(n) \hookrightarrow \mathfrak{so}(2n)$$

and for the other classical Lie algebras, we find

$$\begin{aligned} \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) &\hookrightarrow \mathfrak{sl}(m \cdot n), \\ \mathfrak{so}(m) \oplus \mathfrak{so}(n) &\hookrightarrow \mathfrak{so}(m \cdot n), \\ \mathfrak{sp}(m) \oplus \mathfrak{sp}(n) &\hookrightarrow \mathfrak{so}(4m \cdot n), \\ \mathfrak{so}(m) \oplus \mathfrak{Sp}(n) &\hookrightarrow \mathfrak{Sp}(m \cdot n) \text{ and} \\ \mathfrak{so}(2m+1) \oplus \mathfrak{so}(2n+1) &\hookrightarrow \mathfrak{so}(2m+2n+2). \end{aligned}$$

For the exceptional's, things are more complicated and one finds:

| \mathfrak{g} | maximal special subalgebras |
|----------------|---|
| E_6 | $A_2^{[5]}$, $G_2^{[3]}$, $C_4^{[1]}$, $F_4^{[1]}$, $A_2^{[2]} \oplus G_2^{[1]}$ — Dynkin index of embedding |
| E_7 | $A_1^{[231]}$, $A_1^{[359]}$, $A_2^{[21]}$, $A_1^{[15]} \oplus A_1^{[24]}$, $A_1^{[7]} \oplus G_2^{[2]}$, $A_1^{[3]} \oplus F_4^{[1]}$, $C_3 \oplus G_2^{[1]}$ |
| E_8 | $A_1^{[520]}$, $A_1^{[760]}$, $A_1^{[1240]}$, $B_2^{[12]}$, $A_1^{[16]} \oplus A_2^{[6]}$, $F_4^{[1]} \oplus G_2^{[1]}$ |
| F_4 | $A_1^{[156]}$, $A_1^{[8]} \oplus G_2^{[1]}$ |
| G_2 | $A_1^{[28]}$ |

12.4. Branching rules

remember: In physics Lie algebras do not appear directly but through representations.

Question: How do irreps of a Lie algebra \mathfrak{g} decompose into irreps of its subalgebra \mathfrak{h} ?

Answer: Encoded in branching rules

Take V_Λ a module of \mathfrak{g} with highest weight Λ and V_{λ_i} modules of \mathfrak{h} — " — λ_i

then $V_\Lambda(\mathfrak{g}) \rightarrow \bigoplus_j V_{\lambda_j}(\mathfrak{h})$ and in terms of characters

$$\chi_\Lambda^{(\mathfrak{g})} = \sum_j \chi_{\lambda_j}^{(\mathfrak{h})}$$

In particular: $\dim V_\Lambda = \sum_j \dim V_{\lambda_j}$

To obtain branching rules we need

- 1) projection matrix $P_{\mathfrak{g} \rightarrow \mathfrak{h}}$ relating the weights of \mathfrak{g} to the weights of the subalgebra \mathfrak{h}
- 2) after applying $P_{\mathfrak{g} \rightarrow \mathfrak{h}}$, we use the same technique

