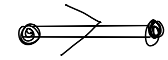
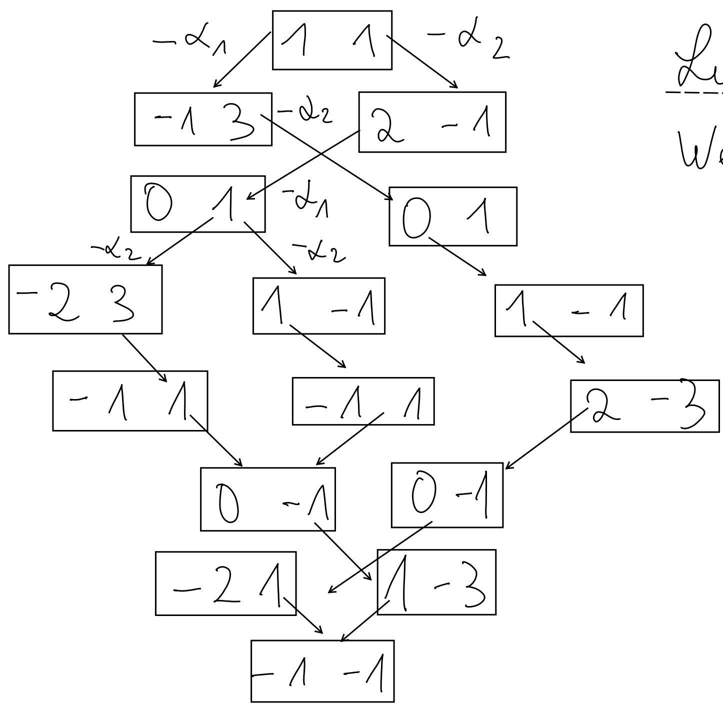
last lecture: highest weight irreps, i.e. $\boxed{1\ 1}$
in $so(5) = B_2 =$ 

Cartan matrix $A_{ij} = \begin{pmatrix} \boxed{2 \ \ -2} \\ \boxed{-1 \ \ 2} \end{pmatrix} \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix}$



Lie ART:

WeightSystem[Irrep[B][1,1], SpindleShape $\to$ True]

- multiplicity: here, count how often weight appears, in general Freudenthal red. formula
- dimension: count weights $\circ$ multipl.

$= 16$ //

11.1. Characters

Question: Is there an easier way to obtain the dimension of a highest weight irrep?

Answer: Weyl character formula!

Def.: The character of a module V is the mapping of the Cartan subalgebra g_0 to $\mathbb{C}$ defined by
$$\chi_V(\mu) := \mathrm{tr}_V\left(e^{\mu^i \rho(H_i)}\right).$$

Due to the trace in this definition, we in particular find:

1) $\chi_{V_1 \oplus V_2}(\mu) = \chi_{V_1}(\mu) + \chi_{V_2}(\mu)$,

2) $\chi_{V_1 \otimes V_2}(\mu) = \chi_{V_1}(\mu) \cdot \chi_{V_2}(\mu)$

Remember that we decomposed the irreducible module V into weights $V = \bigoplus_{\{\lambda\}} V_\lambda$

therefore we find: $\chi_V(\mu) = \sum_\lambda \text{mult}_V(\lambda)\, e^{(\mu,\lambda)}$

$\rightarrow \chi_V(\mu)$ charaterises an highest weight module uniquely

Example: su(2) $\chi_\Lambda(\mu) = \sum_\lambda e^{(\lambda,\mu)} = \sum_{n=0}^{\Lambda} e^{\mu(\Lambda-2n)}$

$\text{mult}_\Lambda(\lambda) = 1$

$$= \sum_{n=0}^{\Lambda} e^{\mu\Lambda}\left(e^{-2\mu}\right)^n \underset{\text{geometric series}}{=} e^{\mu\Lambda}\, \frac{1-\left(e^{-2\mu}\right)^{\Lambda+1}}{1-e^{-2\mu}}$$

$$= \frac{\sinh[\mu(\Lambda+1)]}{\sinh \mu}$$

check: $\lim_{\mu\to 0} \chi_\Lambda(\mu) = 1+\Lambda = \dim V_\lambda$

and $2 \otimes 2 = 1 \oplus 3$

$\chi_1(\mu)\cdot\chi_1(\mu) = \chi_0(\mu) + \chi_2(\mu)$

For the general case: **Weyl character formular**

$$\boxed{\chi_\Lambda(\mu) = \frac{\sum_{w\in W} \text{sign}(w)\, e^{(w(\Lambda+\rho),\mu)}}{\sum_{w\in W} \text{sign}(w)\, e^{(w(\rho),\mu)}}}$$

remember $\rho = (\underbrace{1,\dots,1}_{\text{rank } g})$

Weyl group, later

+ denominator identity

$$\sum_{w\in W} \text{sign}(w)\, e^{(w(\rho),\mu)} = \prod_{\alpha>0}\left[e^{1/2(\alpha,\mu)} - e^{-1/2(\alpha,\mu)}\right]$$

all positive roots

we evaluate the character at $\mu = 0$ to get the

$$\boxed{\underline{\text{Weyl dimension formula}} \quad \dim(V_\Lambda) = \mathrm{tr}_{V_\Lambda}(1) = \chi_\Lambda(0) = \prod_{\alpha>0} \frac{(\Lambda+\rho,\alpha)}{(\rho,\alpha)}}$$

11.2. Weyl group

Remember, the root system Φ usually has discrete symmetries, we denote by $\mathrm{Aut}(\Phi) \subset S_{\dim g - \mathrm{rank}\, g}$ automorphism (structure preserving map to itself)

A particular subgroup is the <u>Weyl group $W(g)$</u>. It is generated by

$$\begin{aligned} w_\alpha : \beta \mapsto w_\alpha(\beta) &:= \beta - (\beta, \alpha^\vee)\alpha \\ &= \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \ , \quad \beta \in \Phi \end{aligned}$$

w_α is called <u>Weyl reflection</u>.

One can show that w_α:

1) maps indeed roots to roots
2) maps even simple roots to simple roots
3) leaves the inner product of roots invariant

The <u>Weyl group</u> arises by composition of w_α's.
→ not all its elements have to be Weyl reflections themself. It acts

1) transitively (any basis of simple roots can be obtained from a given one by a suitable $w \in W(g)$)
2) and freely (this transformation is unique)