Quanten Field Theory Summer 2022

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13. Quantisation of General Relativity and String Theory

To be discussed on Tuesday, 14th June, 2022 in the seminar.

In the last lecture, we discussed some of the problems which one encounters while trying to quantise the classical theory governing gravity, general relativity. A central ingredient in this work has been some fundamental concepts from differential geometry, like a metric compatible covariant derivative and the Riemann tensor it gives rise to. In this exercise, we will discuss them by looking at a simple example, a two sphere.
Moreover, for those of you how would like to improve their grades in the tutorials, there is an extra problem about the one loop beta-functions of the two-dimensional non-linear σ-model, which describes a closed string. It is worth six extra points.

Exercise 13.1: Differential geometry of a 2-sphere

Consider the metric of a 2-sphere of radius a:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2 \left[d\theta^2 + \sin^2\theta d\phi^2 \right] .$$

The metric encodes all information on the geometry of the manifold. We will determine all geometric quantities that are relevant for general relativity:

a) **The metric:** Choosing $x^1 = \theta$ and $x^2 = \phi$, read of the matrix $g_{\mu\nu}$. Show that you obtain this metric by embedding a sphere in $\mathbb{R}^3$ with

$$x^1 = a\cos\theta , \qquad x^2 = a \sin\theta\cos\phi , \qquad \text{and} \qquad x^3 = a\sin\theta\sin\phi .$$

b) **The Christoffel symbols:** The Christoffel symbols are defined as

$$\Gamma^\kappa_{\lambda\mu} = \frac{1}{2} g^{\kappa\nu} \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} \right) .$$

They enter covariant derivatives such as $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$, where the correction term with the Christoffel symbols ensures that the covariant derivative indeed transforms "covariantly" under arbitrary coordinate transformations $x^\mu \to x'^\mu(x^\nu)$, i.e.,

$$\nabla_\mu V^\nu \to (\nabla_\mu V^\nu)' = \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\rho} \nabla_\lambda V^\rho ,$$

without second derivatives in the coordinates.
Compute the non-vanishing Christoffel symbols for the two-sphere. *Hint:* $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, *so only a few components have to be computed explicitly.*

c) **The Riemann tensor:** The Riemann curvature tensor has the form

$$R^\kappa_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\eta_{\nu\lambda}\Gamma^\kappa_{\mu\eta} - \Gamma^\eta_{\mu\lambda}\Gamma^\kappa_{\nu\eta} .$$

Calculate the non-vanishing components of $R^\kappa_{\lambda\mu\nu}$ for the two-sphere *Hint: Use the anti-symmetry in* μ *and* ν *to avoid redundant computations.*

d) **The Ricci tensor:** The Ricci tensor is defined as

$$\mathrm{Ric}_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \,.$$

Calculate $\mathrm{Ric}_{\mu\nu}$ for S^2.

e) **The scalar curvature:** The scalar curvature is given as

$$\mathcal{R} = g^{\mu\nu}\mathrm{Ric}_{\mu\nu} \,.$$

Calculate $\mathcal{R}$ for S^2. How does the scalar curvature behave in the limit $a \to \infty$? Interpret this behaviour.

f) **The Einstein tensor:** The Einstein equation is the field equation of general relativity. It relates the curvature of spacetime to the matter distribution:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \,,$$

where G denotes Newton's constant, $T_{\mu\nu}$ is the energy momentum tensor and $G_{\mu\nu}$ denotes the Einstein tensor:

$$G_{\mu\nu} = \mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} \,.$$

Calculate $G_{\mu\nu}$ for S^2.

Exercise 13.2: One-loop β-function of 2D non-linear σ-model

Note: By solving this exercise, you can get six extra points for the tutorials.

Derive the one loop β-functions for the non-linear σ-model

$$S = \frac{1}{4\pi\alpha'} \oint \mathrm{d}^2\sigma \, G_{ij}\partial_\mu X^i \partial^\mu X^j$$

in full detail by using the background field method, we explained in the lecture. The challenge in this problem, and this is what you get the points for, is to understand every small step of the derivation. I do not just want you to copy derivations from a book or lecture notes. You should understand the steps and be able to explain every detail.