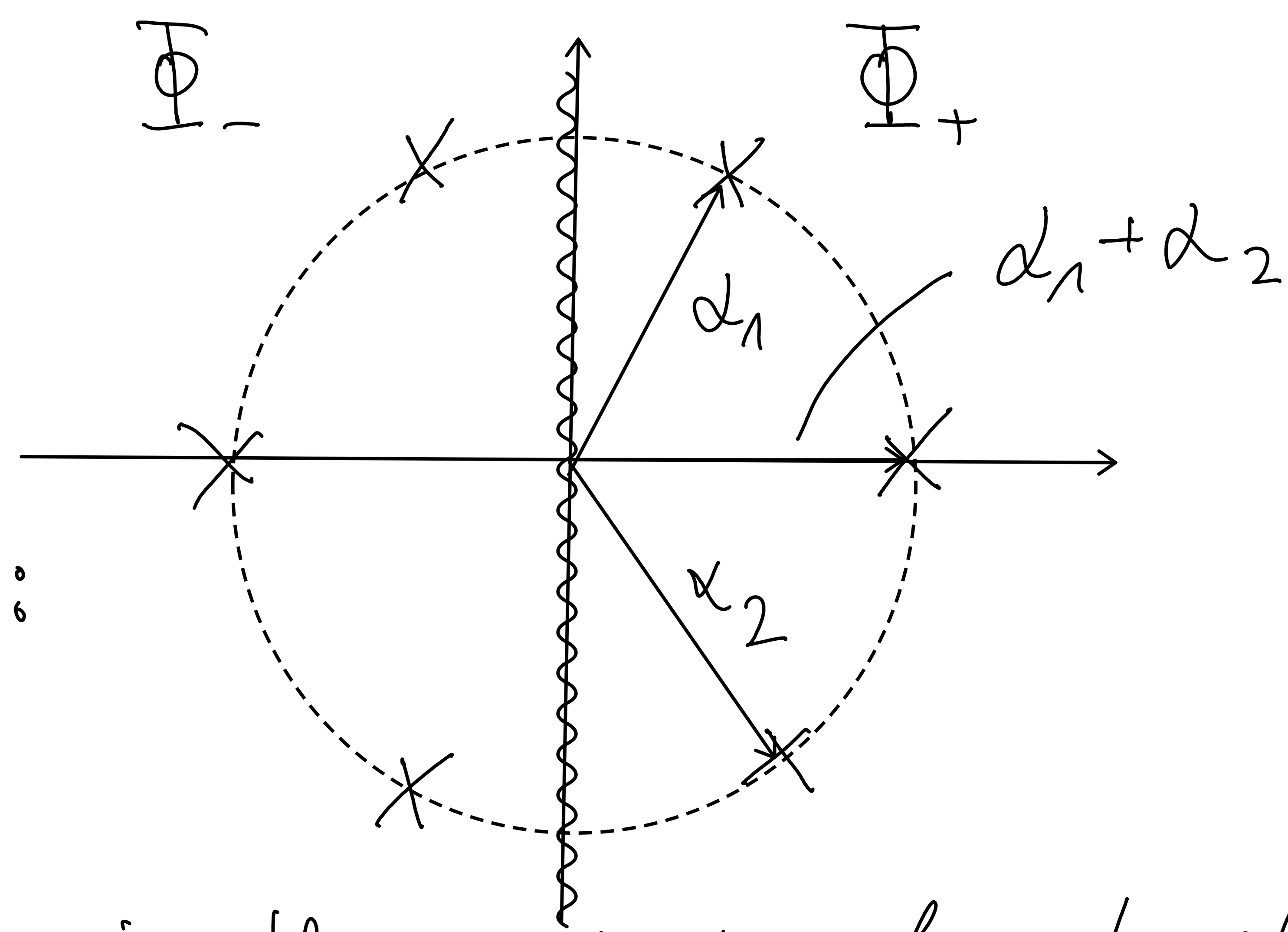
6. Simple Roots and Cartan Matrices

6.1. Simple Roots

last lecture: We introduced the root system of a simple Lie algebra.

i.e. $su(3)$
(here in adapted basis for the Cartan generators):



Question: What is the minimal set of roots we need to find all the others?

Answer: Simple roots!

(I.) find a hyperplane, with no roots on it, that divides the rootsystem into two half spaces

$V_\pm$ ← positive roots / negative roots.

For $\alpha \in V_\pm$, we write $\alpha \gtrless 0$, and split the root system Φ into $\Phi_\pm = \{\alpha \in \Phi \mid \alpha \gtrless 0\}$.

Note: $\alpha \in \Phi_+ \Leftrightarrow -\alpha \in \Phi_-$ and $E_\alpha \hat{=}$ raising operator, $E_{-\alpha} \hat{=}$ lowering operator for $\alpha > 0$

number of elements: $\#(\Phi_\pm) = \dfrac{\dim(g) - \mathrm{rank}(g)}{2} \in \mathbb{N}$

triangular decomposition of g:

$$g = g_+ \oplus g_0 \oplus g_- \;, \quad g_\pm = \mathrm{span}_\mathbb{C}\{E_\alpha \mid \alpha \gtrless 0\}$$

(II) simple roots = all positive roots that cannot be written as of two or more positive roots

Independently of the choice of hypersurface there are $r = \mathrm{rank}\, g$ simple roots. They form a basis of Φ:

$$\mathrm{span}_{\mathbb{R}} \Phi_S = \mathrm{span}_{\mathbb{R}} \Phi \quad \text{with} \quad \Phi_S = \{\alpha_i \mid i = 1, \dots, r\}$$

any pos. root = sum of simple roots with non-neg. coefficients

remember Cartan-Weyl basis:

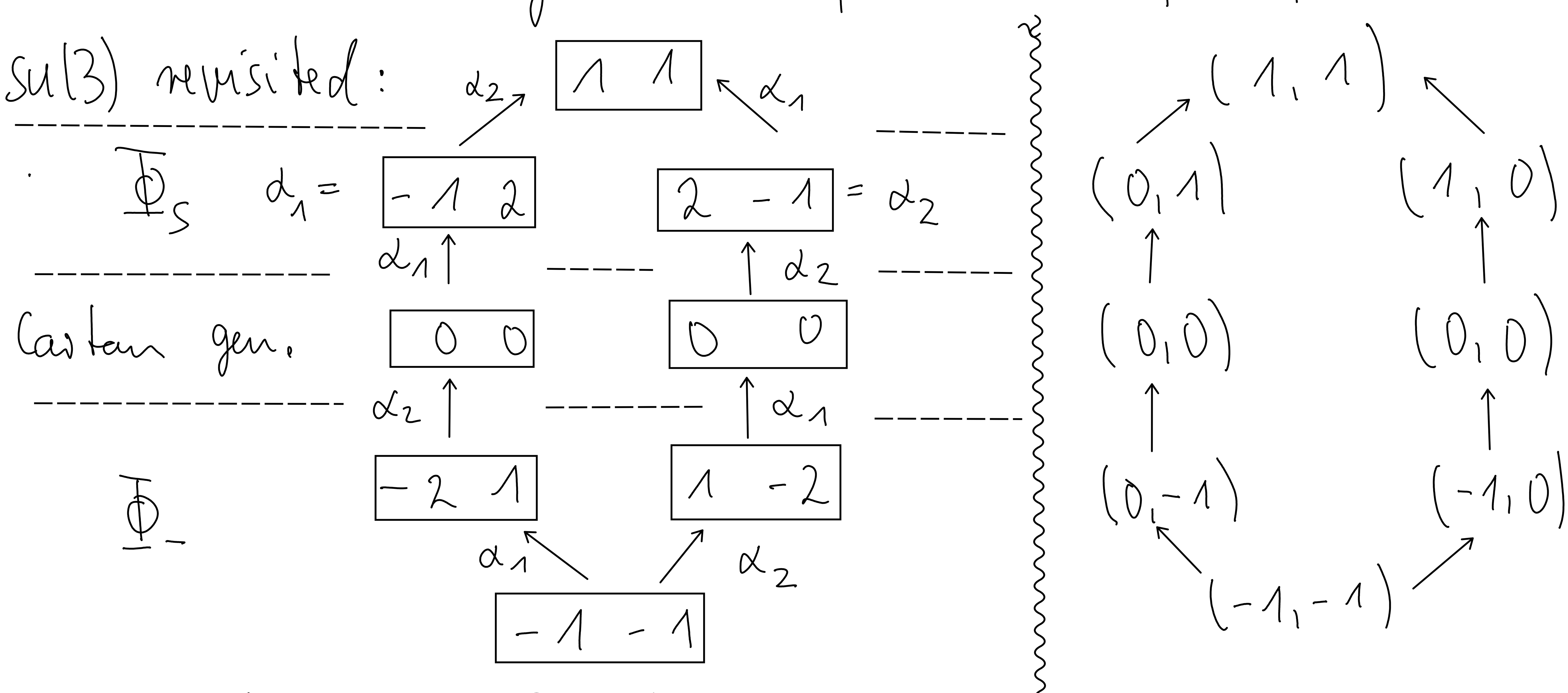
$$[H_i, E_\alpha] = \underset{\substack{\text{\rotatebox{90}{$\le$}}\\ \text{vector}}}{\alpha_i} E_\alpha, \quad [E_\alpha, E_{-\alpha}] = H_\alpha = \alpha^{\vee i} H_i$$

one-form $\hat{=}$ coroot

We can always choose $(\alpha_i)^{\vee j} = \delta_i^j$

such that $H_{\alpha_i} = H_i$.

For an arbitrary coroot $\beta^\vee$ we have $\beta^\vee = \beta^i (\alpha_i)^\vee$.



root basis = Dynkin basis   co-root basis

Lie ART: RootSystem[algebra]   AlphaBasis[root]

i.e. SU3

Question: Relation between root & coroot basis?
Answer: (inverse) Cartan matrix.

6.2. Cartan matrix

Cartan matrix encodes non-orthogonality of the simple roots

$$A_{ij} := (\alpha_i, \alpha_j^\vee) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \qquad i,j = 1, \dots, \operatorname{rank} g$$

Note that:

- rows of A_{ij} are the components of simple roots in Dynkin basis
- NOT always symmetric
- entries are integers

and moreover:

1) $A_{ii} = 2$: follows from definition

2) $A_{ij} = 0 \Rightarrow A_{ji} = 0$: $A_{ij} = 0$ implies $(\alpha_i, \alpha_j) = 0$ because $(\alpha_i, \alpha_i) \neq 0$ and therefore $A_{ji} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = 0$

3) $A_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$

4) The Cartan matrix is non-degenerate.
simple roots form a basis of root space.

5) $A_{ij} A_{ji} \leq 4$ (no sum):
We introduce the angle $\measuredangle(\alpha, \beta)$ between $\alpha, \beta \in \Phi$

$$\cos \measuredangle(\alpha, \beta) = \frac{(\alpha, \beta)}{\sqrt{(\alpha, \alpha)(\beta, \beta)}}$$ (remember scalar product in LAG)

$$A_{ij} A_{ji} = 4 \cos^2 \measuredangle(\alpha_i, \alpha_j) \leq 4$$

Consequence: only 4 possibilities for A_{ij}, A_{ji}, $i \neq j$:

a) $A_{ij} = A_{ji} = 0$ c) $A_{ij} = -2$, $A_{ji} = -1$

b) $A_{ij} = A_{ji} = -1$ d) $A_{ij} = -3$, $A_{ji} = -1$.

for $A_{ij} A_{ji} = 4$ $\measuredangle(\alpha_i, \alpha_j) = 0 \Rightarrow \alpha_i \parallel \alpha_j$ contradicts that α_i and α_j are linear independent.

Block diagonal structure $A = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix}$ is still possible. But then g has proper sub ideals and is semisimple. For simple Lie algebras A is indecomposible.

6.3. Chevalley - Serre Relations

We can now give very compact expressions for the commutators of g in terms of $\{E_{\pm i}, H_i\}$ with $E_{\pm i} = E_{\pm \alpha_i}$. These 3 rank g generators satisfy:

$$\boxed{\begin{array}{ll} 1) & [H_i, H_j] = 0 \\ 2) & [H_i, E_{\pm j}] = \pm A_{ij} E_{\pm j} \\ 3) & [E_{+i}, E_{-j}] = \delta_{ij} H_j \\ 4) & (\mathrm{ad}_{E_{\pm i}})^{1-A_{ij}} E_{\pm j} = 0 \end{array}}$$

Chevalley - Serre relations

$\hat{=}$ How often can I add α_i to α_j before I leave the root system?