An Introduction to String Theory, Winter 2022/23

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3. Polyakov action and its symmetries (21 points)

To be discussed on Thursday, 20th October, 2022 in the tutorial.
Please indicate your preferences until Saturday, 15/10/2022, 21:00:00 on the website.

Exercise 3.1: Field equations of the Polyakov action

Consider the Polyakov action

$$S_{\mathrm{P}} = -\frac{T}{2}\int \mathrm{d}^2\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{1}$$

that we discussed in the lecture.

a) (1 point) In deriving its equations of motion, the identity

$$\det(\exp A) = \exp(\operatorname{Tr} A)$$

will be useful. Derive this relation. Note that $\exp A$ denotes the matrix exponent

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \,.$$

Hint: For our purpose it is sufficient to assume that A can be diagonalised by the similarity transformation $D = SAS^{-1}$.

b) (2 points) Use the just derived relation to show that

$$\delta h = -h_{\alpha\beta}(\delta h^{\alpha\beta}) h \,,$$

where $h = \det(h_{\alpha\beta})$.

c) (1 point) The energy momentum tensor $T_{\alpha\beta}$ describes the response of the action to changes in the metric:

$$\delta S = -T \int \mathrm{d}^2\sigma \sqrt{-h}\, T_{\alpha\beta}\delta h^{\alpha\beta} \,.$$

Compute $T_{\alpha\beta}$ for the Polyakov action.

d) (2 points) Find the equations of motion for $h^{\alpha\beta}$ and show that, after some manipulation and reinsertion into (1), one re-obtains the Nambu-Goto action.

e) (2 points) Show that adding a "cosmological constant term",

$$S_1 = \Lambda \int \mathrm{d}^2\sigma \sqrt{-h} \,,$$

to the Polyakov action leads to inconsistent field equations for $h_{\alpha\beta}$ in the combined szstem $S_{\mathrm{P}} + S_1$ when $\Lambda \neq 0$.

Exercise 3.2: Continuous symmetries

a) (1 point) Show that the Weyl invariance,

$$S_{\mathrm{P}}[e^{2\lambda}h_{\alpha\beta}, X^\mu] = S_{\mathrm{P}}[h_{\alpha\beta}, X^\mu]\,,$$

of (1) automatically implies $h^{\alpha\beta}T_{\alpha\beta} = 0$ without the use of the equations of motion.

b) (2 points) Verify the tracelessness of $T_{\alpha\beta}$ directly by using your results for $T_{\alpha\beta}$ from 1c).

c) (2 points) How does $h_{\alpha\beta}$ has to transform under arbitrary reparameterisations

$$(\tau, \sigma) \to (\tilde{\tau}(\tau, \sigma), \tilde{\sigma}(\tau, \sigma))$$

for (1) to be invariant.

Exercise 3.3: Poincaré symmetry

Consider Lorentz rotations and translations acting on the target space as,

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu\,.$$

a) (1 point) Show that the Polyakov action (1) is invariant under (3).

b) (2 points) Write all infinitesimal generators of the transformation (3).
Hint: You should find D translations (they are the easiest to obtain) and $D(D-1)/2$ Lorentz rotations. See the lecture notes for some further hints.

c) (2 points) Find the conserved changes corresponding to all of them.

d) (3 points) Calculate the Poisson brackets of these changes and show that they are governed by the Poincaré algebra.