Tutorial for String Theory I, WiSe2013/14
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8. Path integrals, ghosts and Grassmann numbers

To be discussed on Thursday, December 12, 2013 in the tutorial.

Exercise 8.1: Gaussian integrals

In the following, integration over $\mathbb{R}^n$ is always understood.

a) Compute

$$Z(0) := \int d^n\vec{x} e^{-\frac{1}{2}\vec{x}^T A \vec{x}}, \quad A^T = A .$$

b) Compute

$$Z(\vec{j}) := \int d^n x e^{-\frac{1}{2}\vec{x}^T A \vec{x} + \vec{j}^T \vec{x}}, \quad \vec{j} \in \mathbb{R}^n .$$

c) Define

$$\langle B(\vec{x}) \rangle := \frac{1}{Z(0)} \int d^n\vec{x} B(\vec{x}) e^{-\frac{1}{2}\vec{x}^T A \vec{x}} .$$

Show that $\langle 1 \rangle = 1$.

d) Show that

$$\int d^n\vec{x} \frac{\partial}{\partial x^i} \left(B(\vec{x}) e^{-\frac{1}{2}\vec{x}^T A \vec{x}} \right) = 0 .$$

e) Show that

$$\langle B(x_1, \ldots, x_n) \rangle = \left. \frac{B\left(\frac{\partial}{\partial j_1}, \ldots, \frac{\partial}{\partial j_n}\right) Z(\vec{j})}{Z(0)} \right|_{\vec{j}=0}$$

f) Compute $\langle x_i x_j \rangle$:

i) directly,

ii) using d) and taking $B(\vec{x}) = x_j$,

iii) using e).

g) Compute $\langle x_i x_j x_k x_l \rangle$:

i) using d) with $B(\vec{x}) = x_j x_k x_l$,

ii) using e).

Exercise 8.2: Faddeev-Popov ghosts and Grassmann numbers

Determinants of operators such as the Faddeev-Popov determinant $\Delta_{\mathrm{FP}} = \det P$ can formally be written as a separate path integral over a now set of auxiliary variables. In order for this to be possible, these auxiliary variables have to be anti-commuting rather than ordinary commuting numbers. Two anti-commuting numbers (or Grassmann numbers) ϕ and η satisfy

$$\phi\eta = -\eta\phi$$

and hence $\phi^2 = 0$. Because of this, the most general function of on Grassmann variable ϕ is

$$f(\phi) = A + B\phi$$

with A, $B \in \mathbb{C}$.
Integrals over Grassmann variables ("Berezin integrals") are defined by

$$\int d\phi [A + B\phi] := B\,. \tag{1}$$

a) Defining the derivative

$$\frac{d}{d\phi}\phi = 1\,, \quad \frac{d}{d\phi}A = 0 \quad (A \in \mathbb{C})\,,$$

show that the Berezin integral of a total derivative is zero and that the Berezin integral is translation invariant, i.e.,

$$\int d\phi \frac{d}{d\phi} f(\phi) = 0$$
$$\int d\phi f(\phi + a) = \int d\phi f(\phi) \quad \text{for } a \in \mathbb{C}\,.$$

These properties mimic similar properties of ordinary integrals of the type $\int\limits_{-\infty}^{\infty} dx f(x)$, which is the motivation for the unusual definition (1). Note that, for Grassmann variables, integration and differentiation are equivalent operations.

b) If one has several linearly independent Grassmann variables ϕ_i ($i = 1, \ldots, n$), where

$$\phi_i\phi_j = -\phi_j\phi_i \quad \forall\, i\,, j\,,$$

one defines

$$\int d\phi_1 \ldots d\phi_n f(\phi_i) = c\,,$$

where c is the coefficient in front of the $\phi_n\phi_{n-1}\ldots\phi_1$-term in $f(\phi^i)$ (note the order):

$$f = \cdots + c\phi_n\phi_{n-1}\ldots\phi_1\,.$$

Let n be even and split the ϕ_i into two sets ψ_m, χ_m ($m = 1, \ldots, \frac{n}{n}$):

$$(\phi_1, \ldots \phi_n) = (\psi_1, \chi_1, \psi_2, \chi_2, \ldots, \psi_{\frac{n}{2}}, \chi_{\frac{n}{2}})\,.$$

Show that

$$\Big(\prod_{m=1}^{\frac{n}{2}} \int d\psi_m d\xi_m\Big) e^{\sum\limits_{m=1}^{\frac{n}{2}} \chi_m\lambda_m\psi_m} = \prod_{m=1}^{\frac{n}{2}} \lambda_m\,,$$

where $\lambda_m \in \mathbb{C}$ are ordinary c-numbers and the exponential is defined via its power series expansion.

c) If the λ_m are the eigenvalues of an operator Λ, one thus obtains

$$\Big(\prod_{m=1}^{\frac{n}{2}} \int d\psi_m d\chi_m\Big) e^{\sum\limits_{m,l=1}^{\frac{n}{2}} \chi_m \Lambda_{ml} \psi_l} = \det \Lambda \,,$$

or, in a path integral context with Grassmann-valued fields $\psi(x)$, $\chi(x)$ and a differential operator Δ,

$$\int \mathcal{D}[\psi]\mathcal{D}[\chi] e^{\int d^d x \chi \Delta \psi} = \det \Delta \,.$$

Using similar arguments (see, e.g. Polchinski, Chapter 3.3 for a detailed account), one obtains

$$\det P = \int \mathcal{D}[c_\alpha]\mathcal{D}[b^{\beta\gamma}] \exp\left[-\frac{i}{4\pi}\int d^2\sigma \sqrt{h} b^{\alpha\beta}(Pc)_{\alpha\beta}\right] \,,$$

where $b^{\alpha\beta}(\sigma) = \beta^{\beta\alpha}(\sigma)$ is a symmetric traceless anti-commuting field, and $c_\alpha(\sigma)$ is an anti-commuting world sheet vector field. Show that, due to the symmetry and tracelessness of $b^{\alpha\beta}$, one can write

$$\det P = \int \mathcal{D}[c_\alpha]\mathcal{D}[b^{\beta\gamma}] \exp\left[-\frac{i}{2\pi}\int d^2\sigma \sqrt{h} b^{\alpha\beta}\nabla_\alpha c_\beta\right] \,.$$

d) It is more convenient to use $b_{\alpha\beta}$ ("anti-ghost") and c^α ("ghost") as the independent fields, as they turn out to be neutral under Weyl transformations, whereas $b^{\alpha\beta}$ and c_α are not due to additional powers of the (inverse) metric. Use

$$S_{\text{ghost}} = -\frac{i}{2\pi}\int d^2\sigma \sqrt{h} b_{\alpha\beta}\nabla^\alpha c^\beta$$

to derive the ghost action in flat world sheet light cone coordinates:

$$S_{\text{ghost}} = \frac{i}{\pi}\int d^2\sigma \left(c^+\partial_- b_{++} + c^-\partial_+ b_{--}\right) \,. \tag{2}$$

e) Derive the equations of motion for $c^\pm$ and $b_{\pm\pm}$ from (2).

f) The total gauge fixed path integral is now

$$Z = \int \mathcal{D}[X]\mathcal{D}[c]\mathcal{D}[b] e^{i[S_{\text{P}}+S_{\text{ghost}}]h_{\alpha\beta}=\eta_{\alpha\beta}} \,,$$

and one clearly sees that it would have been inconsistent to simply set $h_{\alpha\beta} = \eta_{\alpha\beta}$ and drop the $\mathcal{D}[h]$ integration, as that would have missed the ghost contribution. To appreciate the ghost contribution, one notes that the total energy momentum tensor $T_{\alpha\beta}$ now also gets a contribution from the ghost action

$$T_{\alpha\beta} = T_{\alpha\beta}[X] + T_{\alpha\beta}[b,c]$$

which modifies the central charge term in the Virasoro algebra to

$$A(m) = \frac{D}{12}m(m^2-1) + \frac{1}{6}(m - 13m^3) + 2am \,.$$

A non-vanishing total $A(m)$ translates to an anomaly of the local Weyl transformations. Verify that this anomaly is absent if and only if $D = 26$ and $a = 1$.