4. Non-abelian gauge symmetries & Lie groups

last lecture: reminder on topics of your QED course

today: push these ideas to learn something new

4.1. Particles & global symmetries

Idea: find more general local symmetries (other than the $U(1)$ we encountered in QED)

i.e. two Dirac fermions:

$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ which transform as

$$\Psi' = \underbrace{\exp\left(i \frac{\alpha^i \sigma_i}{2} \right)}_{V} \Psi$$

2x2 Pauli matrices, mix the two spinors ψ_1 and ψ_2

$\sigma_i \otimes \mathbb{1}_4$

$\sigma_i^+ = \sigma_i$ implies $V^+ = V^{-1}$

unitary matrix hints that we will work with $SU(2)$

(I) global symmetry:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial})\Psi - m\bar{\Psi}\Psi \qquad \bar{\Psi} = (\psi_1^+\gamma_0 \quad \psi_2^+\gamma_0)$$

$$\bar{\Psi}' = \Psi'^+\gamma_0 = \Psi^+ V^+ \gamma_0 = \bar{\Psi} V^+$$

(V^+ and γ_0) commute

$\mathbb{1}_2 \otimes \gamma_0$

(II) conserved currents:

$$\delta\Psi = \alpha^i \frac{\partial}{\partial \alpha^i} V \Big|_{\alpha^j = 0} \Psi = i \frac{\alpha^i \sigma_i}{2} \Psi$$

$$\delta S = 0 = \ldots = \int d^4x \, \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \Psi)} \delta\Psi(\alpha^i) \right) = \int d^4x \, \partial_\mu J^\mu_i \alpha^i$$

use the field equations (for "…"); J^μ_i: conserved current

$$= \int d^4x \, \partial_\mu \left(-\frac{\alpha^i}{2} \bar{\Psi} \gamma^\mu \sigma_i \Psi \right)$$

$$\boxed{J^\mu_i = -\frac{1}{2} \bar{\Psi} \gamma^\mu \sigma_i \Psi} \qquad \text{with} \quad \partial_\mu J^\mu_i = 0$$

conserved charges $\quad Q_i = \int d^3x \, J^0_i$

⚡ What is the interpretation of Q_i?

remember canonical momentum: $\pi = \frac{\delta \mathcal{L}}{\delta \dot{\psi}} = i\psi^+$ $\quad \{\ \gamma_0^2 = 1$

$$Q_i = \int d^3x \, \frac{(i)^2}{2} \psi^+ \sigma_i \psi = \frac{i}{2} \int d^3x \, \pi \sigma_i \psi$$

$$\{Q_i, Q_j\} = -1/4 \int d^3x \int d^3y \, \{\pi \sigma_i \psi(\vec{x}), \pi \sigma_j \psi(\vec{y})\}$$

Poisson brackets $= \dots$

$$= -\frac{i}{4} \int d^3x \, \pi \underbrace{[\sigma_i, \sigma_j]}_{= 2i\varepsilon_{ij}{}^{k}\sigma_k} \psi(\vec{x})$$

$$\{Q_i, Q_j\} = i\varepsilon_{ij}{}^{k} Q_k$$

$\downarrow$ canonical quantisation

$[Q_i, Q_j] = i\varepsilon_{ij}{}^{k} Q_k \rightarrow$ compare with angular momentum

$$[L_i, L_j] = i\varepsilon_{ij}{}^{k} L_k$$

<u>and remember:</u>

casimir operator $\qquad$ spans the cartan subalgebra

We can only diagonalise $L^2 = \sum_i L_i^2$ and L_3 at the same time because $[L^2, L_3] = 0$

$$L^2 \, |l \, m\rangle = l(l+1) \, |l \, m\rangle \qquad l = 0, 1/2, 1, \dots$$
$$L_3 \, |l \, m\rangle = m \, |l \, m\rangle \qquad m = -l, -l+1, \dots, l$$

spin quantum number

$|1 \; 1\rangle$

$|1/2 \; 1/2\rangle$

$|0 \; 0\rangle$ $\qquad$ $|1 \; 0\rangle$

singlet

$|1/2 \; -1/2\rangle$

$|1 \; -1\rangle$

doublet

triplet

analog: eigenvalues of Q_3 = iso spin

$N_u \dots$ number of up quarks $\qquad Q_3 = \frac{1}{2}(N_u - N_d)$

$N_d \dots$ number of down quarks

<u>Doublet</u> = fundamental representation of the Lie algebra $su(2)$

all other reps arise from tensor products

$$\underline{\text{i.e}} \qquad |1/2 \;\; 1/2\rangle = u$$
one quark $\qquad |1/2 \;\; -1/2\rangle = d$

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two quarks, spin = 0

$$|1 \;\; 1\rangle = u\bar{d} \qquad \pi^+$$
$$|1 \;\; 0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \qquad \pi^0$$
$$|1 \;\; -1\rangle = d\bar{u} \qquad \pi^-$$

$\left.\right\}$ mesons

$$|0 \;\; 0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$\nearrow$ EX 3.2 for details

$\Rightarrow$ $\boxed{\text{particles organise into representations of the global symmetry!}}$

Idea of Yang & Mills <u>gauge</u> this symmetry
turn global into local!

4.2. The Yang-Mills Lagrangian

To make the global $su(2)$ symmetry local, we need a <u>covariant derivative</u>!

connection or gauge field

Remember $U(1)$ case: $\quad D_\mu \phi = (\partial_\mu + ie A_\mu)\phi$

now: $\quad \boxed{D_\mu = \partial_\mu - ig A^i_\mu \frac{\sigma_i}{2}}$

<u>check</u>: $\quad \delta D_\mu \psi = \partial_\mu \delta\psi - ig\, \delta A^i_\mu \frac{\sigma_i}{2}\psi - ig A^i_\mu \frac{\sigma_i}{2}\delta\psi$

$$= i\alpha^i \frac{\sigma_i}{2} D_\mu \psi \underbrace{- \frac{1}{4} g \alpha^i A^j_\mu [\sigma_i, \sigma_j]\psi + i\partial_\mu \alpha^i \frac{\sigma_i}{2}\psi - ig\, \delta A^i_\mu \frac{\sigma_i}{2}\psi}_{\stackrel{!}{=}\, 0}$$

$$\delta A^i_\mu \frac{\sigma_i}{2} = \frac{1}{g}\partial_\mu \alpha^i \frac{\sigma_i}{2} + \frac{i}{4} g\alpha^i A^j_\mu [\sigma_i, \sigma_j]$$

<u>Field strength</u>: $\quad [D_\mu, D_\nu]\psi = -ig F^i_{\mu\nu} \frac{\sigma_i}{2}\psi$

$$\ldots \quad F^i_{\mu\nu}\frac{\sigma_i}{2} = 2\partial_{[\mu} A^i_{\nu]} \frac{\sigma_i}{2} - ig A^i_\mu A^j_\nu \left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right]$$

simplify by using: $\quad t_i := \frac{\sigma_i}{2}$

$$[t_i, t_j] = \overset{\sqrt{-1}}{i} f_{ij}{}^k t_k$$

results in: $$\boxed{\begin{aligned} \delta A_\mu^i &= g^{-1}\partial_\mu \alpha^i - g\,\alpha^j A_\mu^k f_{jk}{}^i \\ F_{\mu\nu}^i &= 2\partial_{[\mu} A^i_{\nu]} + g A_\mu^j A_\nu^k f_{jk}{}^i \end{aligned}}$$

Remarks: – holds not just for SU(2) but for any Lie group
– for $f_{ij}{}^k = 0$, we find the results from last lecture

Lagrangian: • should not transform → trivial representation

Killing metric: $\boxed{2\,\mathrm{Tr}(t_i t_j) = K_{ij}}$ $\begin{cases} 1 & \text{for } i=j \\ 0 & \text{otherwise} \end{cases}$

compute: $2\,\mathrm{Tr}\left(\frac{\sigma_i}{2}\frac{\sigma_j}{2}\right) = \frac{1}{2}\mathrm{Tr}(\sigma_i\sigma_j) = \delta_{ij} = K_{ij}$

K^{ij} is the inverse with $K^{ik}K_{kj} = \delta^i_j$

use to raise & lower "Lie algebra indices"

i.e. $F_{i\mu\nu} = K_{ij} F^j_{\mu\nu}$ $\{ g_{\mu\nu}\partial^\nu = \partial_\mu$

now the most general Lagrangian with two derivatives is

$$\boxed{\mathcal{L} = \bar{\Psi}(i\not{D})\Psi - \tfrac{1}{4} F_{i\mu\nu}F^{i\mu\nu} - c\,\varepsilon^{\mu\nu\sigma s} F_{i\mu\nu}F^i{}_{\sigma s} - m\Psi\bar{\Psi}}$$

comment: – third term violates discret P and T symmetry
parity time reversal

$P: x^i \to -x^i$

$T: x^0 \to -x^0$

$\Rightarrow c = 0$ (see discussion about the QCD θ-angle)

4.3 The standard model

gauge groups realised in nature:

$SU(3) \times \underbrace{SU(2) \times U(1)}$ ← broken by Higgs mechanism

quantum chromo dynamic electroweak interaction

~ strong force ~ electro magnetic & ~ weak for