An Introduction to String Theory, Winter 2022/23

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4. Riemannian geometry and Virasoro algebra (21 points)

To be discussed on Thursday, 27th October, 2022 in the tutorial.
Please indicate your preferences until Saturday, 22/10/2022, 21:00:00 on the website.

Exercise 4.1: Fun with the covariant derivative

Some of you might not have encountered Lie and covariant derivatives in a course on general relativity or differential geometry. Therefore, we will here prove some of their most important properties. Remember, that the covariant derivative ∇_μ acts as

$$\begin{aligned}(\nabla_\mu T)^{\nu_1\dots\nu_r}{}_{\rho_1\dots\rho_s} = \partial_\mu T^{\nu_1\dots\nu_r}{}_{\rho_1\dots\rho_s} + \Gamma^{\nu_1}{}_{\sigma\mu} T^{\sigma\nu_2\dots\nu_r}{}_{\rho_1\dots\rho_s} + \Gamma^{\nu_r}{}_{\sigma\mu} T^{\nu_1\dots\nu_{r-1}\sigma}{}_{\rho_1\dots\rho_s} \\ - \Gamma^{\sigma}{}_{\rho_1\mu} T^{\nu_1\dots\nu_s}{}_{\sigma\rho_2\dots\rho_s} - \Gamma^{\sigma}{}_{\rho_s\mu} T^{\nu_1\dots\nu_s}{}_{\rho_1\dots\rho_{s-1}\sigma}\,,\end{aligned} \tag{1}$$

where $\Gamma^\mu{}_{\nu\rho}$ denote the Christoffel symbols (of second kind)

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho}\right) \tag{2}$$

have to be computed from the metric $g_{\mu\nu}$.

Hint: Keep in mind that the Christoffel symbols are symmetric with respect to the two lowered indices.

a) (1 point) Show that

$$\nabla_\mu g_{\nu\rho} = 0$$

holds. We call this property metric compatibility.

Hint: Use the definitions (1) *and* (2) *to compute the right hand side explicitly.*

b) (1 point) Show that you can substitute the partial derivatives in the Lie derivative for a vector,

$$L_\xi v^\mu = \xi^\nu \partial_\nu v^\mu - v^\nu \partial_\nu \xi^\mu\,,$$

for the covariant derivative and still get the same rule.

Hint: For this to work it is crucial to keep in mind that the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}$ are symmetric in ν and ρ.

c) (1 point) Remember from the lecture that the metric transforms as

$$\delta g_{\mu\nu} = L_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho(\mu}\partial_{\nu)}\xi^\rho\,.$$

Use this rule to obtain the transformation of the Christoffel symbols.

Hint: Keep in mind that the variation δ commutes with the partial derivative ∂, i.e. $\delta(\partial_\mu \dots) = \partial_\mu(\delta \dots)$.

d) (2 points) Using the result from c) and

$$\delta v^\mu = L_\xi v^\mu\,,$$

show that

$$\delta(\nabla_\mu v^\nu) = L_\xi(\nabla_\mu v^\nu)$$

holds.

e) (2 points) Compute the Riemann tensor by evaluation using

$$[\nabla_\mu, \nabla_\nu] v^\rho = R_{\mu\nu\sigma}{}^\rho v^\sigma \,. \tag{3}$$

Exercise 4.2: Differential geometry of the two-sphere

Consider the metric of a 2-sphere of radius a

$$ds^2 = R^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \right) \,.$$

we derived in the first exercise lecture. This metric encodes all information on the geometry of the manifold and allows us to determine all geometric quantities that are relevant for general relativity. Please compute explicitly for this metric:

Hint: You can check your results by using the Mathematica package GREAT (General Relativity, Einstein & All That).

a) (2 points) **The Christoffel symbols**, as defined in (2) .

Hint: Again take into account that they are symmetric with respect to the two lowered indices, so only a few components have to be computed explicitly.

b) (2 points) **The Riemann tensor**, which has the from

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\eta{}_{\nu\lambda}\Gamma^\kappa{}_{\mu\eta} - \Gamma^\eta{}_{\mu\lambda}\Gamma^\kappa{}_{\nu\eta} \,.$$

Hint: Use the antisymmetry in μ and ν to avoid redundant computations.

c) (1 point) **The Ricci tensor and the scalar curvature**, which are given by

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \qquad \text{and} \qquad R = g^{\mu\nu} R_{\mu\nu} \,.$$

How does the scalar curvature behave in the limit $R \to \infty$? Interpret this behaviour.

d) (1 point) **The Einstein tensor**, which is part of the field equations of the Einstein-Hilbert action. It relates the curvature of space and time to the matter distribution given in terms of the energy-momentum tensor:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{5}$$

where G denotes Newton's constant, $T_{\mu\nu}$ is the energy momentum tensor and $G_{\mu\nu}$ denotes the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,.$$

Exercise 4.3: The classical Virasoro algebra

In the lecture we learned about the classical Virasoro algebra and discussed the Poisson brackets of its generating, conserved charges. Now, lets derive the algebra from first principles.

a) (3 points) Using light-cone coordinates and conformal gauge with the energy-momentum tensor

$$T_{\pm\pm} = \frac{1}{2} \partial_\pm X^\mu \partial_\pm X_\mu$$

and the canonical Poisson brackets

$$\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = \{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\} = 0$$
$$\{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\} = \frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma')\,,$$

calculate the Poisson brackets

$$\{T_{\pm\pm}(\sigma,\tau), X^\mu(\sigma',\tau)\}\,.$$

b) (2 points) The definition

$$L_\xi := 2T\int_0^{2\pi} \mathrm{d}\sigma\xi(\sigma^+)T_{++}(\sigma^+)$$

form the lecture and the result from a) to calculate the Poisson bracket

$$\{L_\xi, X^\mu(\sigma,\tau)\}$$

and show that the L_ξ generate infinitesimal conformal transformations via the Poisson bracket.

c) (2 points) Expand the function $\xi(\sigma^+)$ into Fourier components $e^{im\sigma^+}$. The corresponding charges than generate the classical Virasoro algebra with respect to the Poisson bracket, namely

$$\{L_m, L_n\} = -i(m-n)L_{m+n}\,.$$

Verify that expression and check. We want to interpret the Poisson bracket as a Lie bracket. Check to this end if the Jacobi identity is satisfied.

d) (1 point) Show that the generators L_0, L_{-1} and L_1 form a Lie subalgebra.