Tutorial for String Theory I, WiSe2013/14
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# 11. String compactification on the circle

To be discussed on Thursday, January 16, 2014 in the tutorial.

**Exercise 11.1: The bosonic string on $S^1$: The specturm**

As was shown in lecture, the sector with vanishing Kaluza-Klein momentum and zero winding number (i.e., the sector with $M = L = 0$), simply corresponds to the ordinary field theoretic zero-modes of the 26D fields when they are dimensionally reduced on the circle. In particular, the sector with $N = \bar{N} = 0$ describes the 25D remnant of the 26D tachyon, whereas the sector with $N = \bar{N} = 1$ describes the 25D massless fields that descend from the massless 26D fields $G_{MN}$ (the metric), $B_{MN}$ (the Kalb-Ramond field) and $\Phi$ (the dilaton). Under this dimensional reduction, the 26D metric $G_{MN}$ decomposes into the 25 metric $G_{\mu\nu}$ ($\mu, \nu, \dots = 0, 1, 2, \dots, 24$), a 25D vector field $G_{\mu,25}$ and a 25D scalar $G_{25,25}$, whereas the two-form $B_{MN}$ gives rise to a 25D two-form $B_{\mu\nu}$ and a 25D vector $B_{\mu,25}$, and the dilaton $\Phi$ simply leads to a 25D scalar. The vacuum expectation value $\langle G_{25,25}\rangle \sim R$ of the scalar $G_{25,25}$ describes the (dynamically undetermined) size of the circle, and the two vector fields gauge two U(1)'s.
A prime example for truly stringy states without a point particle analogue, on the other hand, is given by the states with $(M, L) = (\pm1, \pm1)$ and $(M, L) = (\pm1, \mp1)$. For these states, the level matching condition

$$N - \bar{N} = ML$$

implies $N - \bar{N} = 1$ and $N - \bar{N} = -1$, respectively. The lowest lying modes in this sector correspond to $(N, \bar{N}) = (1, 0)$ and $(N, \bar{N}) = (0, 1)$, respectively. As was shown in lecture, each of these two cases leads to two 25D vector fields and two 25D scalars with a radius dependent mass

$$m^2_{\mathrm{25D}} = \frac{1}{R^2} + \frac{R^2}{4} - 1 \geq 0\,.$$

For $R = \sqrt{2} = \sqrt{\alpha'}$, these states become massless, and the four vector fields combine with $G_{\mu,25}$ and $B_{\mu,25}$ to fill out the adjoint representation of $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$, which is Higgsed to $\mathrm{U}(1)_{\mathrm{L}}\times\mathrm{U}(1)_{\mathrm{R}}$ by the scalar field $G_{25,25}$ at generic values $R \neq \sqrt{\alpha'}$.
In this exercise, we take a closer look at some of the other states in the spectrum that were not yet discussed in the lecture.

a) Show that in the sectors with $(M, L) = (\pm1, \pm1)$ and $(M, L) = (\pm1, \mp1)$ and $(N \cdot \bar{N}) > 0$, there can be no massless or tachyonic states, no matter how the radius $R$ is chosen.

b) Consider now the sector with $|ML| > 1$. Can there be states in this sector that can become massless or tachyonic at some particular values $R$? If yes, give the corresponding values of $R$.

c) Consider now the sector $M \neq 0$ and $L = 0$. What is the constraint on the occupation numbers $N$ and $\bar{N}$ for these states?

d) Show that for any given $M \neq 0$ with $L = 0$ and $N = \bar{N} = 0$, all three cases (tachyonic, massless, massive) can be realized by choosing $R$ appropriately. What is the mass squared value for the special radius $R = \sqrt{2} = \sqrt{\alpha'}$?

e) Show that, for $M \neq 0$, $L = 0$ and $N = \bar{N} \geq 1$ ($R < \infty$), there can only be massive states.

f) Repeat parts c) through e) for states of the form $M = 0$, $L \neq 0$.

**Exercise 11.2: Charges of Kaluza-Klein and winding modes**

As was shown in lecture, the compactification of the closed string on a circle of radius $R$ leads to two 25D vector fields $G_{\mu,25}$ and $B_{\mu,25}$ that are massless for arbitrary values of the radius $R$. These two vector fields correspond to zero modes of the $(\mu, 25)$-components $(\mu, \nu, \dots = 0, 1, \dots, 24)$ of the 26D metric $G_{MN}$ and the 26D two-form field $B_{MN}$ $(M, N, \dots = 0, 1, \dots, 25)$, respectively. They give rise to a gauge group $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ that remains unbroken for all $R$. In addition, the closed string spectrum also contains four more vector fields that can also become massless, but only at the self-dual radius $R = \sqrt{2} = \sqrt{\alpha'}$. They correspond to the excitations

$$\begin{aligned} |V^{\mu}_{\pm}\rangle &= \alpha^{\mu}_{-1}|M = \pm 1, L = \pm 1\rangle \\ |V'^{\mu}_{\pm}\rangle &= \bar{\alpha}^{\mu}_{-1}|M = \pm 1, L = \mp 1\rangle\,. \end{aligned}$$

In the lecture, it was claimed that these additional vector fields combine with $G_{\mu,25}$ and $B_{\mu,25}$ to fill out the full adjoint representation of $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$, which should thus be considered as the full gauge group, which is unbroken at the self-dual radius, but Higgsed to $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ at generic values of $R$. For this to be possible, $V^{\mu}_{\pm}$ and $V'^{\mu}_{\pm}$ have to be charged with respect to the U(1)'s gauged by $G_{\mu,25}$ and $B_{\mu,25}$ (just as the $W^{\pm}$-bosons have to be charged in the Standard Model). In this exercise, we will uncover the physical origin of this charge.

To understand this origin, we have to understand first how the closed string couples to the metric $G_{MN}$ and the two-form $B_{MN}$. So far, we have only studied the propagation of strings in flat Minkowski spacetime (corresponding to $G_{MN} = \eta_{MN}$) and without any background two-form field $B_{MN}$ turned on. The action in this simplified case is just the Polyakov action,

$$\begin{aligned} S_{\mathrm{P}} &= -\frac{T}{2}\int d^2\sigma \sqrt{h} h^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \eta^{MN} \\ &\overset{h_{\alpha\beta}=\eta_{\alpha\beta}}{=} -\frac{T}{2}\int d^2\sigma \left(-\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N\right)\eta_{MN}\,. \end{aligned}$$

In a more general, curved, background with metric $G_{MN}(X)$, this is simply generalized by replacing the constant Minkowski metric $\eta_{MN}$ by the curved metric $G_{MN}(X)$:

$$S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma \left(-\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N\right) G_{MN}(X(\sigma,\tau))\,. \tag{1}$$

The coupling of a string to a non-vanishing background $B_{MN}$-field, on the other hand, is described by adding the action

$$\begin{aligned} S_{\mathrm{B}} &= \frac{T}{2}\int d^2\sigma \epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N B_{MN}(X(\sigma,\tau)) \\ &= T\int d\sigma d\tau \partial_\tau X^M \partial_\sigma X^N B_{MN}(X(\sigma,\tau))\,, \end{aligned} \tag{2}$$

where $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$, $(\epsilon^{\tau\sigma} = 1)$ is the 2D epsilon tensor. This action is simply the integral of the pull-back of the two-form $B_{MN}$ to the two dimensional world sheet and can be viewed as a higher-dimensional analogue of the eletromagnetic coupling of a point particle of charge $q$ in $d$ dimensions with worldline $x^\mu(\tau)$:

$$S_{\mathrm{e.m.}} = \int d^d x j^\mu A_\mu = q\int d\tau (\partial_\tau x^\mu) A_\mu(x(\tau))\,, \tag{3}$$

with $j^\mu(x) = \int d\tau q \dot{x}^\mu \delta^{(d)}(x^\mu - x^\mu(\tau))$ being the current density. The integrand here is likewise nothing but the pull-back of the one-form $A_\mu(x)$ to the worldline of the particle.

a) Consider the mode expansions ($\alpha' = 2$)

$$\begin{aligned} X^\mu(\sigma,\tau) &= x^\mu + 2p^\mu\tau \quad (+\text{oscillators}) \\ &\equiv x^\mu(\tau) \quad (+\text{oscillators}) \end{aligned} \tag{4}$$

$$X^{25} = x^{25} + 2\frac{M}{R}\tau + LR\sigma \quad (+\text{oscillators})\,. \tag{5}$$

Setting all oscillators terms in (4) and (5) equal to zero and considering only constant[1] $G_{MN}(x^\mu, x^{25}) = G_{MN}$, calculate the term in (1) that is proportional to $G_{\mu,25}$, and compare this with $S_{\text{e.m.}}$ in (3) to infer that the charge of a string with respect to the 25D vector field $G_{\mu,25}$ is proportional to its Kaluza-Klein momentum number $M$. Does the winding number $L$ also enter the charge, and if yes, what is the proportionality?

b) Make a similar analysis for the action $S_{\text{B}}$ in equation (2) and show that the charge of a string without oscillators with respect to the (constant or "$\sigma$-averaged" part of the) 25D vector field $B_{\mu,25}$ is proportional to the winding number $L$. What is the dependence upon the Kaluza-Klein momentum number $M$ in this case?

**Conclusions:** Due to the non-trivial Kaluza-Klein momentum numbers $M$ and winding numbers $L$, the states $|V^\mu_\pm\rangle$ and $V'^\mu_\pm\rangle$ are charged with respect to (linear combinations of) $B_{\mu,25}$ and $G_{\mu,25}$, as they should in order to fit in th adjoint of SU(2) groups.

### Exercise 11.3: T-duality

In a circle compactification for the coordinate $X^{25}$, the T-duality transformation acts on the coordinate field $X^{25}(\sigma,\tau) = X^{25}_{\text{L}}(\tau+\sigma) + X^{25}_{\text{R}}(\tau-\sigma)$ as

$$X^{25}(\tau,\sigma) \to \bar{X}^{25}(\sigma,\tau) := X^{25}_{\text{L}}(\tau+\sigma) - X^{25}_{\text{R}}(\tau-\sigma)$$

a) Using the expansion

$$\begin{aligned} X^{25}_{\text{L}}(\tau+\sigma) &= \frac{1}{2}x^{25} + \left(\frac{M}{R} + \frac{1}{2}LR\right)(\tau+\sigma) + \text{oscillators} \\ X^{25}_{\text{R}}(\tau-\sigma) &= \frac{1}{2}x^{25} + \left(\frac{M}{R} - \frac{1}{2}LR\right)(\tau-\sigma) + \text{oscillators}\,, \end{aligned}$$

show that $X^{25}(\sigma,\tau) \to \bar{X}^{25}(\sigma,\tau)$ indeed swaps the rôles of the Kaluza-Klein momentum number $M$ and the winding number $L$.

b) Show that the above $T$-duality leaves the energy momentum tensor $T_{\pm\pm} = \frac{1}{2}\partial_\pm X \cdot \partial_\pm X$ invariant.

c) An open string ca have either Neumann (N) or Dirichlet (D) boundary conditions at each endpoint $\sigma = \sigma^* = 0, \pi$:

$$\begin{aligned} \partial_\sigma X^\mu\big|_{\sigma^*} &= 0 \quad (\text{D}) \\ \partial_\tau X^\mu\big|_{\sigma^*} &= 0 \quad (\text{N}) \end{aligned}$$

Show that the T-duality transformation $X \to \bar{X} = X_{\text{L}} - X_{\text{R}}$ interchanges the two types of boundary conditions.

[1] For a non-constant metric, the relevant vector field is the "$\sigma$-averaged" quantity $\bar{G}_{\mu,25}(\tau) := \int d\sigma G_{\rho,25}(X(\tau,\sigma))$.