3. Classical Matrix Groups

Reminder: ① lecture, we looked at $SO(3)$

② lecture defined some basic concepts:

Lie algebras and (Lie) groups

Today many more examples of groups which are relevant in physics. They are formed (not all!) by matrices that preserve some additional structure $\Rightarrow$ matrix groups.

3.1. General Linear group ($GL(N)$)

we learned: basis of a vector space is <u>not</u> unique

$\rightarrow \quad \{e_i\}, \quad \{e'_i\}, \quad \{e''_i\} \qquad i = 1, \dots, N$

They are related by matrices $A, B, C \in Mat_{N\times N}$

$$e'_i = \sum_j e_j A^j{}_i \,, \qquad e''_i = \sum_j e'_j B^j{}_i \,, \qquad \text{and} \qquad e''_i = \sum_j e_j C^j{}_i$$

The group structure arises by combining them:

$$e''_i = \sum_j e_j C^j{}_i = \sum_j e'_j B^j{}_i = \sum_{j,k} e_k \underbrace{A^k{}_j B^j{}_i}_{\text{matrix multiplication}}$$

$\Rightarrow C = A \cdot B$

$GL(N)$:	$N\times N$ matrices, A, which are invertible, and therefore have $\det A \neq 0$.

Check group structure:

1) $A \cdot B$ is $N\times N$ matrix and $\det(A\cdot B) = \overset{\neq 0}{\det A} \cdot \overset{\neq 0}{\det B} \neq 0$ closure ✓

matrix mult.

2) $A\cdot(B\cdot C) = (A\cdot B)\cdot C$ associative ✓

3) $\mathbb{1} \cdot A = A \cdot \mathbb{1} = A \qquad (\mathbb{1})_i{}^j = \delta_i{}^j$ identity element ✓

$N\times N$ identity matrix

4) $A^{-1}\cdot A = A \cdot A^{-1} = \mathbb{1}$ inverse element ✓

Note: A vector space can be defined for any field $\mathbb{F}$ and so can $GL(N)$ in particular we have: $GL(N,\mathbb{R})$, $GL(N,\mathbb{C})$, and $GL(N,\mathbb{H})$

Representations:

I. covariant vector: $e'_i = \sum_j e_j A^j{}_i$

II. contravariant vector: $V'^i = \sum_j (A^{-1})^i{}_j V^j$

III. „scalar": $\sum_i e'_i v'^i = \sum_{i,j,k} e_j \underbrace{A^j{}_i (A^{-1})^i{}_k}_{(A\cdot A^{-1})^j{}_k = \delta^j{}_k} V^k = \sum_i e_i v^i$

does not transform at all

IV. combine vector(space)s

a) direct sum: $e_i \oplus f_I = (e_i \;\; f_I)$ $\quad e_i \in V_1, \; f_I \in V_2$

$$e'_i \oplus f'_I = \sum_{i\&J} (e_j \;\; f_J) \begin{pmatrix} A^j{}_i & 0 \\ 0 & B^J{}_I \end{pmatrix} = \left(\sum_j e_j A^j{}_i \;\; \sum_J f_J B^J{}_I \right)$$

block diagonal matrices

b) product: $e'_i \otimes f'_I = \sum_{j,J} e_j \otimes f_J \; A^j{}_i \; B^J{}_I$

tensor transformation

Now take $T'_{i_1 i_2} = \sum_{j_1, j_2} T_{j_1 j_2} A^{j_1}{}_{i_1} A^{j_2}{}_{i_2}$

If $T_{i_1 i_2}$ is (anti-) symmetric $T'_{i_1 i_2}$ is, too.

$T_{i_1 i_2} = \pm T_{i_2 i_1}$

$\Rightarrow$ basis of $V^{\otimes 2} = V \otimes V$ splits into

$e_{i_1} \vee e_{i_2} := \frac{1}{2} (e_{i_1} \otimes e_{i_2} + e_{i_2} \otimes e_{i_1})$

and $e_{i_1} \wedge e_{i_2} := \frac{1}{2} (e_{i_1} \otimes e_{i_2} - e_{i_2} \otimes e_{i_1})$

Note in general we have rank-r-tensors in $V^{\otimes r}$ with the basis $\{ e_{i_1} \otimes \ldots \otimes e_{i_r} \}$ which, among others, have totalley (anti-) symmetric contributions

$$e_{i_1} \vee \ldots \vee e_{i_r} := \sum_{\sigma \in S_r} \frac{1}{r!} \sigma (e_{i_1} \otimes \ldots \otimes e_{i_r})$$

$$e_{i_1} \wedge \ldots \wedge e_{i_r} := \sum_{\sigma \in S_r} \frac{1}{r!} (-1)^\sigma \sigma (e_{i_1} \otimes \ldots \otimes e_{i_r})$$

sum over all permutations

$(-1)^\sigma = \begin{cases} +1 & \sigma \text{ even} \\ -1 & \sigma \text{ odd} \end{cases}$

3.2. Volume preserving groups (SL(N))

V vector space over the field $\mathbb{F}$

$\dim V = N$

The element $e_{i_1} \wedge \ldots \wedge e_{i_N}$ of $V^{\otimes N}$ is called volume element.

Standard representation $e_1 \wedge e_2 \wedge \ldots \wedge e_N$ transforms as:

$$e'_1 \wedge \ldots \wedge e'_N = e_1 \wedge \ldots \wedge e_N \cdot (\det A)$$

remember: $\det A := \sum_{j_1,\dots,j_N} \varepsilon_{j_1 \dots j_N} A^{j_1}{}_1 \cdot \ldots \cdot A^{j_N}{}_N$

$$\boxed{\begin{array}{l}SL(N;\mathbb{F}): \text{ Transformations that preserve the volume element,} \\ \quad A \in GL(N,\mathbb{F}) \text{ with } \det A = 1 \text{, form} \\ \quad \text{the \underline{special linear group}.}\end{array}}$$

$\lightning$: for quaternionic matrices A, B

$\det(A \cdot B) \neq \det A \cdot \det B$

$\rightarrow$ we only have $SL(N,\mathbb{R})$ or $SL(N,\mathbb{C})$

$SL(2,\mathbb{C}) = SO(3,1)$
Lorentz group

3.3. Metrices on vector spaces

Def.: A <u>metric</u> or <u>inner product</u> on a vector space V maps two vectors $v, w \in V$ to a number in the associated field $\mathbb{F}$.

$\langle \cdot , \cdot \rangle : V \times V \rightarrow \mathbb{F}$ with the properties:

1) $\langle v, w_1 a + w_2 b \rangle = \langle v, w_1 \rangle a + \langle v, w_2 \rangle b$ linearity in the <u>2nd</u> argument

and either:

2a) $\langle v_1 a + v_2 b , w \rangle = a \langle v_1, w \rangle + b \langle v_2, w \rangle$ linearity in the 1st argument

2b) $\langle v_1 a + v_2 b , w \rangle = a^* \langle v_1, w \rangle + b^* \langle v_2, w \rangle$ anti-linearity in the 1st argument

$\rightarrow$ <u>bilinear</u>
$\rightarrow$ <u>sesquilinear</u> } metrices

metric components are $h_{ij} := \langle e_i , e_j \rangle$

$h_{ij} = \check{h}_{ij} + \hat{h}_{ij}$ with $\check{h}_{ij} = \frac{1}{2}\left(h_{ij} + h_{ji}^{(*)} \right) = \check{h}_{ji}^{(*)}$

symmetric $\quad$ anti-symmetric $\quad \hat{h}_{ij} = \frac{1}{2}\left(h_{ij} - h_{ji}^{(*)} \right) = -\hat{h}_{ji}^{(*)}$

only for sesquilinear metrices

<u>canonical form for h_{ij}</u> (it is always possible to find a basis such that h_{ij} has a particular form)

$\boxed{1.}$ Symmetric sesquilinear: $h^{\vee} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$ $\lambda_1, \dots, \lambda_N \in \mathbb{R}$

real eigenvalues

by rescaling the basis this further simplifies to

$$h^{\vee} = \begin{pmatrix} +\mathbb{1}_{N_+} & 0 & 0 \\ 0 & 0 \cdot \mathbb{1}_{N_0} & 0 \\ 0 & 0 & -\mathbb{1}_{N_-} \end{pmatrix}$$

$N_0 > 0$ make $h^{\vee}$ singular

$\mathbb{1}_N = N \times N$ identity matrix

non-singular metrics $h^{\vee}$ are characterised by (N_+, N_-)

signature of $h^{\vee}$

$\boxed{2.}$ bilinear antisymmetric: Note $(h^{\wedge})^T = -h^{\wedge}$

$\Rightarrow$ $\det h^{\wedge} = \det (h^{\wedge})^T = (-1)^N \det h^{\wedge}$

$\Rightarrow$ $h^{\wedge}$ can only be non-singular when N is even.

Then: $h^{\wedge} = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix}$ and by normalising the basis vectors pairwise:

$$h^{\wedge} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & \\ 0 & 0 & \ddots & \varepsilon \end{pmatrix} = \mathbb{1}_N \otimes \varepsilon \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

3.4. Metric preserving groups

Def.: Metric preserving groups are subgroups of $GL(N, \mathbb{F})$ that preserve a certain inner product:

$e_i \to e'_i = e_j A^j{}_i$ ← Einstein sum convention

$h'_{ij} = \langle e'_i, e'_j \rangle = \langle e_k A^k{}_i, e_l A^l{}_j \rangle = A^{(*)k}{}_i h_{kl} A^l{}_j \overset{!}{=} h_{ij}$

Check if it is a group?

closure requires: $(A^l{}_k B^k{}_i)^{(*)} = B^{(*)k}{}_i A^{(*)l}{}_k$

↯ true for $\mathbb{R}$ and $\mathbb{C}$ but for $\mathbb{H}$ we require complex conjugation: $(q_1 q_2)^* = q_2^* q_1^*$ $q_1, q_2 \in \mathbb{H}$

metric	bilinear	sesquilinear
symmetric	orthogonal $(h_{ij} = h_{ji})$ $O(N_+, N_-; \mathbb{R})$, $O(N_+, N_-; \mathbb{C})$, $\dim_{\mathbb{F}} = \frac{N(N-1)}{2}$ $O(N_+, N_-; \mathbb{H})$	unitary $(h_{ij} = h_{ji}^*)$ $U(N_+, N_-; \mathbb{C})$, $\dim_{\mathbb{R}} = N^2$ $U(N_+, N_-; \mathbb{H})$ $\dim_{\mathbb{R}} = (2N+1)N$
anti-symmetric	symplectic $(h_{ij} = -h_{ji})$ $Sp(2N; \mathbb{R})$, $Sp(2N; \mathbb{C})$, $\dim_{\mathbb{F}} = N(2N+1)$ $Sp(N; \mathbb{H})$ $\dim_{\mathbb{H}} = \frac{N(N+1)}{2}$	for $\mathbb{R}$ $h_{ij} = -h_{ji}^*$ $h'_{ij} = i\, h_{ij}$ $h'_{ij} = h'^*_{ji}$ $\mathbb{C}, \mathbb{H}$

trick treat $\mathbb{H}$ as $\mathbb{C}^2$

3.5. Metric & volume preserving groups

are denoted by an S, for <u>special</u> metric group in front of O, U, Sp: $SO(N_+, N_-)$, $SSp(2N)$ and $SU(N_+, N_-)$