10. Highest weight representations

remember: last lecture all irreps of $su(N)$ with Young tableaux

Question: What about the other simple Lie algebras?
→ today

10.1. Highest weight of $su(2)$

three generators: H, E_+, E_- ← negative root

Cartan generator ↗ ↖ positive root

$$[H, E_\pm] = \pm 2 E_\pm \,, \quad [E_+, E_-] = H \quad \text{(see 6.3.)}$$

Irreps are characterised by highest weight vectors:

V_Λ with $\boxed{\begin{aligned} H V_\Lambda &= \Lambda V_\Lambda \\ E_+ V_\Lambda &= 0 \end{aligned}}$

check: $H E_- V_\Lambda = E_- H V_\Lambda - 2 E_- V_\Lambda = (\Lambda - 2) E_- V_\Lambda$

$$\begin{array}{c} \boxed{V_\Lambda} \\ \downarrow E_- \\ \boxed{V_{\Lambda-2}} \\ \downarrow E_- \\ \vdots \end{array}$$

$$V_{\Lambda-2n} = E_-^n V_\Lambda$$

$$\begin{aligned} E_+ V_{\Lambda-2n} &= E_+ E_- V_{\Lambda-2n+2} \\ &= ([E_+, E_-] + E_- E_+) V_{\Lambda-2n+2} \\ &= (H + \underbrace{E_- E_+}_{\sim V_{\Lambda-2n+2}}) V_{\Lambda-2n+2} \end{aligned}$$

$E_- E_+ V_{\Lambda-2n} = r_n V_{\Lambda-2n}$, therefore

$E_- E_+ V_{\Lambda-2n+2} = r_{n-1} V_{\Lambda-2n+2}$ and

$H V_{\Lambda-2n+2} = (\Lambda - 2n + 2) V_{\Lambda-2n+2}$

E_- on both sides

$$r_n V_{\Lambda-2n} = (\Lambda - 2n + 2 + r_{n-1}) V_{\Lambda-2n}$$

$$r_n = \Lambda - 2n + 2 + r_{n-1} \quad \text{with } r_0 = 0 \quad (E_+ V_\Lambda = 0)$$

Solve with ansatz $r_n = \alpha n^2 + \beta n$

$\alpha = -1,\ \beta = \Lambda + 1:\quad r_n = n(\Lambda + 1 - n)$ and

$r_{\Lambda+1} = 0$

$$\rightarrow \boxed{E_-\, v_{\overbrace{\Lambda - 2\Lambda}^{-\Lambda}} = 0}$$ lowest weight vector

$$\begin{array}{c} \boxed{v_\Lambda} \\ E_+ \uparrow\ \downarrow E_- \\ \boxed{v_{\Lambda-2}} \\ E_+ \uparrow\ \downarrow E_- \\ \vdots\quad\vdots \\ E_+ \uparrow\ \downarrow E_- \\ \boxed{v_{-\Lambda}} \end{array}$$

unitary representations: $E_+^\dagger = E_-$

$\nearrow$ angular momentum in QM:

$H \sim L_z$, $E_+ \sim L_+$, $E_- \sim L_-$ and we also have

$\vec{L}^2 = \frac{1}{4} H(H+2) + E_- E_+$, $\vec{L}^2 v_\Lambda = \frac{1}{4}\Lambda(\Lambda+2) v_\Lambda$

$\hat{=}$ spin $j = \frac{1}{2}\Lambda$

10.2. Highest weight modules

for $su(2)$ we found $V_{(\Lambda)} = \bigoplus_{\lambda=-\Lambda}^{\Lambda} \{x_\lambda v_\lambda \mid x_\lambda \in \mathbb{C}\}$

labeled by Cartan generator's eigenvalues

generalisation: $V = \bigoplus_{\{\lambda\}} V_\lambda$ with $v_\lambda \in V_\lambda$:

$\rho(H_i)\, v_\lambda = \lambda^i v_\lambda$

$\rightarrow$ <u>weight</u> vector $\lambda = (\lambda^1, \ldots, \lambda^r)$, $r = \operatorname{rank} g$

The collection of all weights is called the <u>weight system</u>. It extends the root system, which only contains the weights of the adjoint module.

⚡ Not all V_λ's for generic representations are one-dimensional.

same normalisation as for roots in 6.1.

$\lambda^i = \lambda(H_i) = (\alpha_i^\vee, \lambda) \in \mathbb{Z}$

simple coroots $\nearrow$ with dual <u>fundamental weights</u> Λ_i

$(\Lambda_i, \alpha_j^\vee) = \delta_{ij}$

$\rightarrow$ any weight can be written as

$\lambda = \lambda^i \Lambda_i$ Dynkin labels

Generalisation of su(2) modules:

For any finite dimensional, irreducible module of g there is a highest weight such that $\varrho(E_\alpha) V_\Lambda = 0 \quad \forall \alpha > 0$ (positive roots) where V_Λ is one-dimensional.

- all other weights are obtained by acting with neg. root (lowering operators)

$$\varrho(H_i)\varrho(E_{-\alpha}) V_\lambda = \varrho(H_i E_{-\alpha}) V_\lambda = \varrho([H_i, E_{-\alpha}] + E_{-\alpha} H_i) V_\lambda = [-\alpha^i \varrho(E_{-\alpha}) + \lambda^i \varrho(E_{-\alpha})] V_\lambda$$
$$= (\lambda - \alpha)^i \varrho(E_{-\alpha}) V_\lambda \in V_{\lambda-\alpha}$$

$\rightarrow$ any weight can be written as $\lambda = \Lambda - \beta$,
$\beta = \sum_i n^i \alpha_i, \quad n_i \in \mathbb{N}$

- $\sum_i n^i$ = level or depth of the weight
- If a weight λ appears n-times in this process, it has multiplicity n.

$\mathrm{mult}_\Lambda(\lambda) = n \quad \rightsquigarrow \quad \dim_{\mathbb{C}}(V_\lambda) = n$

$$\mathrm{mult}_\Lambda(\lambda) = \frac{2 \sum_{\alpha>0} \sum_{\substack{m>0 \\ V_{\lambda+m\alpha} \in V_\Lambda}} (\lambda + m\alpha, \alpha)\, \mathrm{mult}_\Lambda(\lambda + m\alpha)}{(\Lambda+\varrho, \Lambda+\varrho) - (\lambda+\varrho, \lambda+\varrho)} \qquad \varrho = \frac{1}{2}\sum_{\alpha>0} \alpha$$

Freudenthal reduction formula

- ~~for~~ $\lambda^i > 0$, we can subtract λ^i-times the root α_i. We stop, when no further α_i's can be substracted.
- The highest weight of the conjugate module $\bar{V}$ is

$\Lambda_{\overline{V}} = -\lambda_{min}$, λ_{min} is the lowest weight of V

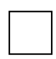
$\longrightarrow$ V is called <u>self conjugate</u> if $\Lambda = -\lambda_{min}$

<u>10.3 Example su(3)</u>

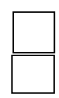
remember : $\alpha_1 = \boxed{2 \;\; -1}$ see 4.7 or

$\alpha_2 = \boxed{-1 \;\; 2}$ $\circ\!-\!\circ$ $A_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$$\Lambda_{(1)} = \begin{tikzcd} \boxed{1 \;\; 0} \arrow[d, "-\alpha_1"] \\ \boxed{-1 \;\; 1} \arrow[d, "-\alpha_2"] \\ \boxed{0 \;\; -1} \end{tikzcd}$$

$\hat{=}\ 3$ or 

$$\Lambda_{(2)} = \begin{tikzcd} \boxed{0 \;\; 1} \arrow[d, "-\alpha_2"] \\ \boxed{1 \;\; -1} \arrow[d, "-\alpha_1"] \\ \boxed{-1 \;\; 0} \end{tikzcd}$$

$\hat{=}\ \overline{3}$ or 

$$\Lambda_{(3)} = \begin{tikzcd} & \boxed{1 \;\; 1} \arrow[dl, "-\alpha_1"'] \arrow[dr, "-\alpha_2"] & \\ \boxed{-1 \;\; 2} \arrow[dr, "-\alpha_2"'] & & \boxed{2 \;\; -1} \arrow[dl, "-\alpha_1"] \\ & \boxed{0 \;\; 0} \arrow[dl, "-\alpha_1"'] \arrow[dr, "-\alpha_2"] & \\ \boxed{-2 \;\; 1} \arrow[dr, "-\alpha_2"'] & & \boxed{1 \;\; -2} \arrow[dl, "-\alpha_1"] \\ & \boxed{-1 \;\; -1} & \end{tikzcd}$$

$\hat{=}\ 8 = 6 + 2$ or