6.4. Quadratic form matrices

fundamental weights

last lecture: Dynkin basis $\boxed{1\ 1}\ \dots$ and
coroot basis $(1,1)$

$$\beta = \beta_i \Lambda^{(i)} \longleftrightarrow \beta^\vee = \beta^i (\alpha_i)^\vee$$

dual, therefore we have an inner product

$$(\alpha,\beta) = \sum_i \alpha_i \beta^i = \sum_i \alpha^i \beta_i = \sum_{i,j} \alpha^i G_{ij} \beta^j = \sum_{i,j} \alpha_i G^{ij} \beta_j$$

$$G_{ij} = \left((\alpha_i)^\vee, (\alpha_j)^\vee\right) = \frac{2}{(\alpha^{(i)},\alpha^{(i)})} A_{ij}$$

important for non-simply-laced g's

$$G^{ij} = (\Lambda^{(i)}, \Lambda^{(j)})$$

$\sum G_{ik} G^{kj} = \delta_i^j$, G_{ij} symmetric, while A_{ij} in general not.

Lie ART: $G^{ij} \mathrel{\hat{=}}$ MetricTensor[algebra]

7. Dynkin diagrams and Classification

7.1. Dynkin diagrams

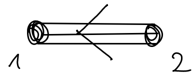
last lecture: Cartan matrix A_{ij} contains all information and is heavily constraint.

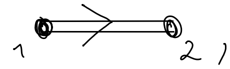
Dynkin diagrams are a compact way to describe A_{ij}.

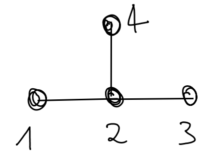
Rules:
1) each simple root is indicated by a node ◎ ;
2) between two nodes one draws $\underbrace{A_{ij} A_{ji}}_{\text{no sum}} \in \{0,1,2,3\}$ lines
3) an arrow points towards the longer root, if both roots are not of the same length.

Examples: – su(3) 

, $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\measuredangle(\alpha^{(1)},\alpha^{(2)}) = 120^\circ$

- g_2  , $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, $\sphericalangle(\alpha^{(1)}, \alpha^{(2)}) = 150°$

- $so(5)$  , $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, $\sphericalangle(\alpha^{(1)}, \alpha^{(2)}) = 135°$

- $so(8)$  , $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$

Question: Does any Dynkin diagram we can imagine give rise to a simple Lie algebra?

Answer: NO! Additional constraints hold.

7.2. Classification of simple Lie algebras

We begin with the Lie algebras of the classical matrix groups from lecture 3.

7.2.1 Classical Lie algebras

$su(N)$: $A + A^\dagger = 0$, $\operatorname{tr} A = 0$, $A \in M_{N\times N}(\mathbb{C})$

using $(\Sigma_{ab})_{cd} = \delta_{ac}\delta_{bd}$ we have:

Cartan generators: $H_i = \Sigma_{ii} - \Sigma_{i+1\, i+1}$,

$i = 1, \dots, r = \operatorname{rank}(su(N)) = N-1$

Simple roots: $E_i = \Sigma_{i\, i+1}$

remember: $[H_i, E_j] = A_{ij} E_j$ and with

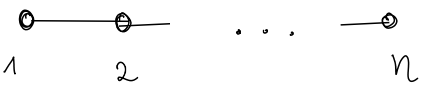
$[\Sigma_{ab}, \Sigma_{cd}] = \delta_{bc}\Sigma_{ad} - \delta_{da}\Sigma_{cb}$ we find

$$[H_i, E_i] = [\Sigma_{ii} - \Sigma_{i+1\, i+1}, \Sigma_{i\, i+1}] = 2\Sigma_{i\, i+1} = 2E_i$$

$$[H_i, E_{i+1}] = [\Sigma_{ii} - \Sigma_{i+1\, i+1}, \Sigma_{i+1\, i+2}] = -\Sigma_{i+1\, i+2} = -E_{i+1}$$

$$[H_i, E_{i-1}] = \dots = -E_{i-1}$$

while all other comm.'s vanish.

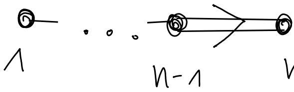
A_n:  $\hat{=}$ su(n+1)

so(2N+1): $A + A^T = 0, \quad A \in M_{2N+1 \times 2N+1}(\mathbb{R})$

$$\mathcal{A}_{ab} = \Sigma_{ab} - \Sigma_{ba} \quad \text{(by construction antisymmetric)}$$

Cartan generators: $H_i = -\underset{=\sqrt{-1}}{i}\, \mathcal{A}_{2i-1\,2i}, \quad i = 1, \ldots, N$

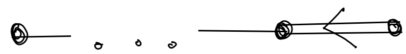
Simple roots:
$$E_i = \begin{cases} \mathcal{A}_{2i-1\,2i+1} + i\mathcal{A}_{2i\,2i+1} - i(\mathcal{A}_{2i-1\,2i+1} + i\mathcal{A}_{2i\,2i+2}), & i < N \\ \mathcal{A}_{2N-1\,2N+1} + i\mathcal{A}_{2N\,2N+1}, & i = N \end{cases}$$

B_n:  $\hat{=}$ so(2n+1, $\mathbb{R}$)

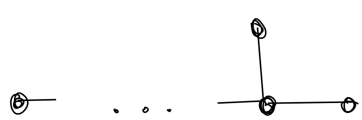
sp(2N): $A^T \varepsilon_{2N} + \varepsilon_{2N} A = 0, \quad A \in M_{2N \times 2N}(\mathbb{R})$

remember $\varepsilon_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

you can work out the details by yourself and get

C_n:  = sp(2n, $\mathbb{R}$)

so(2N): similar to so(2N+1), please check details and verify

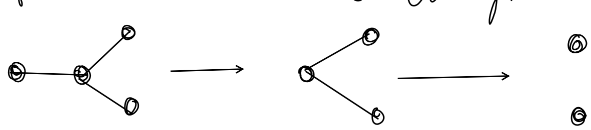
D_n:  = so(2n, $\mathbb{R}$)

7.2.2. Exceptional Lie algebras

Question: Are there more admissable Dynkin diagrams?

$\hat{=}$ produce simple Lie algebra

Rules: 1) Any subdiagram, obtained by removing nodes, of an admissable diagram is also admissable.



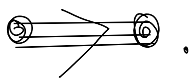
2) There are at most $r-1$ pairs of nodes connected to each other in a rank r simple Lie algebra.

3) There are no diagrams with loops.

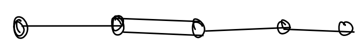
4) No node has more than 3 lines ending on it.

5) Any string of nodes connected by a single line, can be collapsed to a single node, and results in an admissible diagram (the A_n series).

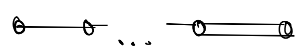
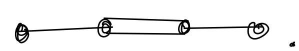
6) There is only one diagram with a triple line:

.

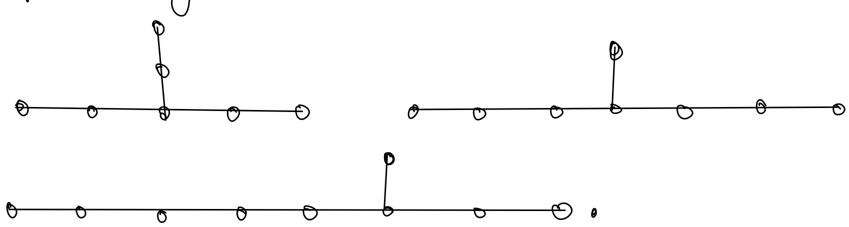
7) Any admissable diagram has at most one double line on a node with 3 single lines ending on it.

8) The diagram  is not admissible.

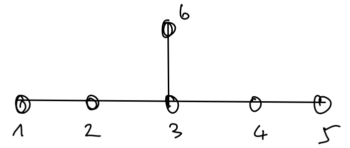
9) The only admissible diagrams with double lines are:

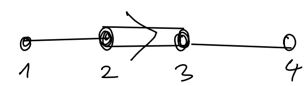
 and .

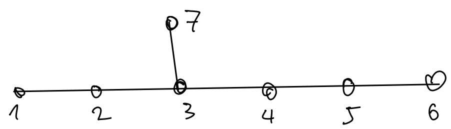
10) Any diagram that can be collapsed to one of the following is not admissible:

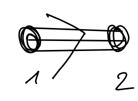
.

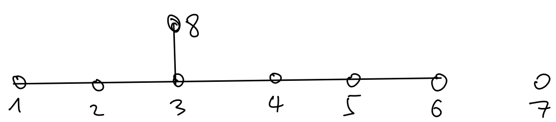
Answer: In addition to the classical A_n, B_n, C_n and D_n series, there are only five more exceptional admissible Dynkin diagrams:

e_6: 

f_4: 

e_7: 

g_2: 

e_8: 

This completes the classification of simple Lie algebras. ☺