Quantum Field Theory

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lectures: Tue. 8:15 – 10:00

seminars: Tue. 10:15 – 12:00

- from 01.03. – 29.03.22 online over MS Teams
- in person in room 447

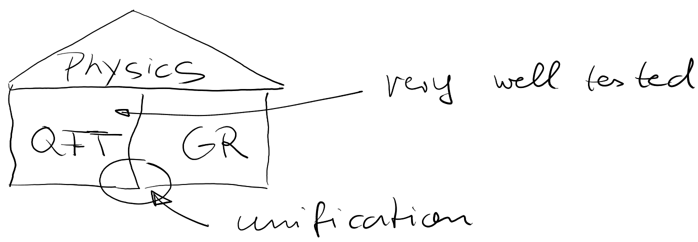
exercises & handwritten notes are posted on the course website:

https://www.fhassler.de/teaching#qft_2022

exercises appear ≈ 1 week before the seminar they are discussed in

exam:
- oral at the end of the semester
- important to attend lectures & seminars
  if you cannot make it, please let me know
- solving the exercise problems is important to pass the exam!

0. Motivation



1. Canonical quantisation

1.1. Klein-Gordon Field: Lagrangian

It is governed by action:

$$S = \int d^4x \, \underbrace{\mathcal{L}}_{\text{Lagrangian}}(\phi, \partial_\mu \phi)$$

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 \qquad \dot{\phi} = \frac{\partial}{\partial t}\phi = \partial_t$$

$$\vec{\nabla} = (\partial_{x^1}, \partial_{x^2}, \partial_{x^3})$$

or better in <u>covariant</u> form with metric

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$\boxed{\mathcal{L} = \frac{1}{2}\underbrace{(\partial_\mu\phi\,\partial^\mu\phi)}_{(\partial_\mu\phi)^2} - \frac{1}{2}m^2\phi^2}$$

equations of motion from <u>principle of least action</u>

$$\delta S = 0$$

$$\delta S = \int d^4x \left[\cancel{\frac{1}{2}}\cancel{2}(\partial_\mu\delta\phi\,\partial^\mu\phi) - \cancel{\frac{1}{2}}\cancel{2}m^2\phi\,\delta\phi \right]$$

derivatives can be "removed" by integration by parts

$$\int d^4x\, \partial_\mu f_1\, f_2 + \int d^4x\, f_1\, \partial_\mu f_2 = \int d^4x\, \partial_\mu(f_1 f_2) = 0$$

we ignore boundary terms at the moment

$$\delta S = \int d^4x\, [-\partial_\mu\partial^\mu\phi - m^2\phi]\,\delta\phi = 0$$

$$\Rightarrow \quad \underbrace{\partial_\mu\partial^\mu}_{\Box}\phi + m^2\phi = 0 \qquad \text{Klein-Gordon equation}$$

1.2 Hamiltonian Theory

<u>classical mechanics</u> variables q^i

$\Updownarrow$

conjugate momentums $p_i = \frac{\partial L}{\partial \dot{q}^i}$

Hamiltonian $H(t) = \sum_i p_i \dot{q}^i - L$

<u>field theory</u> $S = \int dt\, L = \int dt \int d^3x\, \mathcal{L}(\phi, \dot{\phi})$

"variables"

$$\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(x)} \qquad \text{conjugate momentum to field } \phi$$

$$H(t) = \int d^3x\, [\pi \cdot \dot{\phi} - \mathcal{L}] = \int d^3x\, \mathcal{H}$$

integrate over space

Hamiltonian density

For the Klein-Gordon Lagrangian:

$$\Pi(x) = \frac{\delta}{\delta\dot\phi}\left(1/2\,\dot\phi^2 - 1/2(\vec\nabla\phi)^2 - 1/2\,m^2\phi^2\right) = \dot\phi(x)$$

$$H = \int d^3x \left[1/2\,\Pi^2 + 1/2\,(\nabla\phi)^2 + 1/2\,m^2\phi^2\right]$$

Field equations:
1. Hamiltonian ✓
2. Poisson brackets (Pb's)
3. Time evolution

(2.) $\{\phi(t,\vec x), \Pi(t,\vec y)\} = \delta(\vec x - \vec y)$

equal time

all other Pb's are 0

(3.) For all functions of ϕ and π, $\mathcal{O}(\phi,\pi)$, we have

$$\boxed{\frac{\partial}{\partial t}\mathcal{O} = \{\mathcal{O}, H\}}$$

$$\frac{\partial}{\partial t}\phi(t,\vec x) = \int d^3y\, \{\phi(t,\vec x), 1/2\,\Pi^2(t,\vec y)\}$$

$$= \int d^3y \underbrace{\{\phi(t,\vec x), \pi(t,\vec y)\}}_{\delta(\vec x - \vec y)} \pi(t,\vec y)$$

$$= \Pi(t,\vec x)$$

$$\frac{\partial}{\partial t}\Pi(t,\vec x) = \dots = (\vec\nabla^2 - m^2)\,\phi(t,\vec x)$$

$$\ddot\phi = \dot\Pi = (\vec\nabla^2 - m^2)\,\phi$$

$$\ddot\phi - \vec\nabla^2\phi + m^2\phi = \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

Klein Gordon equation

1.3 Quantisation

Fourier transformation to momentum space:

$$\phi(t,\vec x) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec x\vec p}\, \phi(t,\vec p)$$

KG equation: $\left[\frac{\partial^2}{\partial t^2} + \underbrace{(|\vec p|^2 + m^2)}_{\omega^2_{\vec p}}\right]\phi(t,\vec p) = 0$

$$\omega_{\vec p} = \sqrt{|\vec p| + m^2}$$

$\hookrightarrow$ Harmonic oscillators with frequency $\omega_{\vec{p}}$

first on HO

$$H_{HO} = \frac{1}{2} p^2 + \frac{1}{2}\omega^2 \phi^2$$

$$\phi = \frac{1}{\sqrt{2\omega}} (a + a^+) \qquad p = -i\sqrt{\frac{\omega}{2}}(a - a^+)$$

$[\phi, p] = i\underbrace{\hbar}_{=1}$ implies $[a, a^+] = 1$

raising op.

lowering op.

$a|0\rangle = 0$ vacuum or ground state

$$(a^+)^n |0\rangle = |n\rangle$$

$$H_{HO} = \omega(a^+a + 1/2) \qquad a^+a|n\rangle = n|n\rangle$$

$$\omega(n + 1/2)|n\rangle = H_{HO}|n\rangle$$

In the KG theory

$$\phi(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}(t) + a^+_{-\vec{p}}(t)\right) e^{i\vec{p}\cdot\vec{x}}$$

usually suppress t

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} - a^+_{-\vec{p}}\right) e^{i\vec{p}\vec{x}}$$

$$[a_{\vec{p}}, a^+_{\vec{p}}] = (2\pi)^3 \delta(\vec{p} - \vec{p}\,')$$

$\hookrightarrow$ $[\phi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y})$ what you should

compare with $\{\phi(\vec{x}), \pi(\vec{y})\} = \delta(\vec{x} - \vec{y})$

canonical quantisation

$$\boxed{\{\cdot,\cdot\} \rightarrow i\underbrace{\hbar}_{=1 \text{ for us}}[\cdot,\cdot]}$$

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a^+_{\vec{p}} a_p + \underbrace{\tfrac{1}{2}[a_{\vec{p}}, a^+_{\vec{p}}]}_{\infty}\right)$$

∞ = vacuum energy

ignore it here

Spectrum:

$$a_{\vec p} \, |0\rangle = 0$$ vacuum state

$a^+_{\vec p} \, |0\rangle$ = 1-particle state with momentum $\vec p$ and energy $E_{\vec p} = \omega_{\vec p} = \sqrt{|\vec p|^2 + m^2}$

remember $c = 1$ (speed of light)

1.4. Heisenberg picture & Propagator

$$\phi(x) = e^{iHt} \, \phi(\vec x) \, e^{-iHt}$$

4-position $x^\mu = (\underset{= t}{x^0}, \underbrace{x^1, x^2, x^3}_{\vec x})$

same for $\pi(x)$

now we have: $i \frac{\partial}{\partial t} \mathcal{O} = [\mathcal{O}, H]$

compare with $\frac{\partial}{\partial t} \mathcal{O} = \{\mathcal{O}, H\}$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(a_{\vec p} \, e^{-ip_\mu x^\mu} + a^+_{\vec p} \, e^{ip_\mu x^\mu} \right) \Big|_{p^0 = E_{\vec p}}$$

on shell

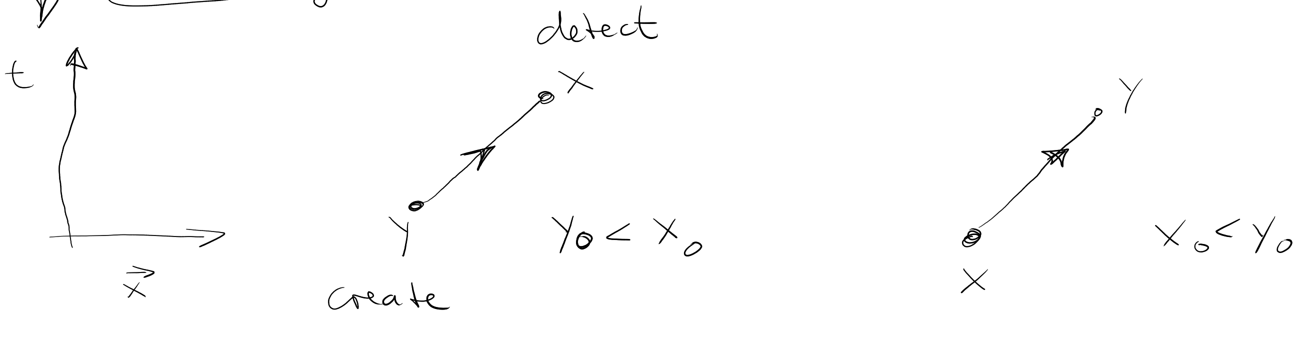
$$\pi(x) = \frac{\partial}{\partial t} \phi(x)$$

Propagator:

experiment: create particle at position y and detect it (annihilate) at position x

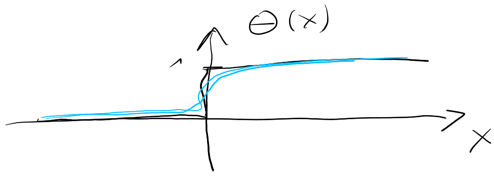
$$D(x-y) = \langle 0 | \, \phi(x) \, \phi(y) \, | 0 \rangle$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-ip(x-y)}$$

⚡ Causality



$$D_F(x-y) = \theta(x^0-y^0)\langle 0|\phi(x)\,\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\,\phi(x)|0\rangle$$

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$



$$= \lim_{\varepsilon\to 0^+} \mp \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\tau \pm i\varepsilon}\, e^{\mp i x \tau}\, d\tau$$

$$\boxed{D_F(x-y) = \lim_{\varepsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\, e^{-ip(x-y)}}$$ Feynman Propagator

↓ Ex 1.1.

$$D_F(x-y) = \theta(x^0-y^0)\,D(x-y) + \theta(y^0-x^0)\,D(y-x)$$
$$= \langle 0|T\,\phi(x)\,\phi(y)|0\rangle$$

time ordering symbol: later operators go to the left

2. Dirac Field

2.1. Lorentz transformations

coordinates: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ — element of the Lorentz group $O(3,1)$

such that $x^\mu x^\nu g_{\mu\nu} = x'^\mu x'^\nu g_{\mu\nu}$

scalar fields: $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$

vector fields: $V^\mu(x) \to V'^\mu(x) = \Lambda^\mu{}_\nu V'^\nu(\Lambda^{-1}x)$

example: $V^\mu(x) = \partial^\mu\phi(x)$

dual field "one-form": $A_\mu(x) \to A'_\mu(x) = (\Lambda^{-1})^\nu{}_\mu A_\nu(\Lambda^{-1}x)$

$A_\mu V^\mu$ is a scalar

example: $A_\mu(x) = \partial_\mu\phi(x)$

Question:
- Are there more examples? Yes there are!
- How do we classify them?

↗ Monographic lecture Lie algebras & Lie groups

so(3,1) Lie algebra

$\frac{1}{2}\, 4\cdot(4-1) = 6$ generators $\qquad J^{\mu\nu} = -J^{\nu\mu}$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)$$

- $J_1 = J^{23}$, $J_2 = J^{31}$, $J_3 = J^{12}$ gen. rotations of the 3 spacial directions
- $K_1 = J^{01}$, $K_2 = J^{02}$, $K_3 = J^{03}$ boosts

$[J_i, J_j] = i \sum_{k=1}^{3} \varepsilon_{ijk} J_k = i\varepsilon_{ijk} J_k$ $\qquad$ so(3) Lie algebra $\subset$ so(3,1)

$[J_i, K_j] = i\varepsilon_{ijk} K_k$

$[K_i, K_j] = -i\varepsilon_{ijk} J_k$

Task: find explicit representations for $J^{\mu\nu}$!

2.2. γ-matrices and the Dirac algebra

More concret: find 4 $n\times n$ matrices γ^μ with

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\cdot \mathbb{1}_{n\times n}$$

$\mathbb{1}_{n\times n}$: $n\times n$ identity matrix

then $\boxed{J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$

EX. 1.2. check that they indeed generate so(3,1) !

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \sigma^i = \text{Pauli matrices}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we also need one more γ-matrix:

$$\underline{\underline{\gamma^5}} = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_{2\times2} & 0 \\ 0 & \mathbb{1}_{2\times2} \end{pmatrix} \quad \text{with } \{\gamma^5, \gamma^\mu\} = 0$$

and therefore: $[\gamma^5, J^{\mu\nu}] = 0$

γ-matrices act on 4-component vectors

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

called <u>Dirac-spinors</u>.

They decompose into 2 fundamental irreps of so(3,1):
the 2-component (but complex) Weyl spinors ψ_L & ψ_R.
They are the ∓ 1 eigenspaces of γ^5.

Finally we need to contract two Dirac-spinors to get a Lorentz scalar (transforms trivially)

naively $\psi^\dagger \psi$ does not work !

but $\bar{\psi} \psi$ with: $\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$

Dirac conjugate

EX 1.3 verify that $\bar{\psi}\psi$ is a Lorentz scalar!

2.3 Dirac equation

$$\boxed{S_{Dirac} = \int d^4x \; \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi}$$

notation $\gamma^\mu \partial_\mu = \not{\partial}$
or $\gamma^\mu p_\mu = \not{p}$

field equations: $\dfrac{\delta S_{Dirac}}{\delta \bar{\psi}} = 0$

$\boxed{(i\not{\partial} - m)\psi = 0}$ Dirac equation

Plane wave solutions

$$\psi(x) = u(p)\, e^{-ipx} + v(p)\, e^{ipx}, \quad p^0 > 0$$

$\to$

$$(\not{p} - m)\, u(p) = 0$$
$$(-\not{p} - m)\, v(p) = 0$$

two linearly independent solutions for

$u(p) = u^s(p)$ $s = 1,2$ and $v(p) = v^r(p)$ $r = 1,2$

which can be normalised to

$$\bar{u}^r(p)\, u^s(p) = 2m\, \delta^{rs}$$
$$\bar{v}^r(p)\, v^s(p) = -2m\, \delta^{rs}$$
$$\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0$$

2.4. Quantisation of the Dirac Field

$\{\gamma^0, \gamma^0\} = 2\gamma^0\gamma^0 = 2 g^{00} \mathbb{1}$

$\gamma^0\gamma^0 = \mathbb{1}$

$\bar{\Psi} = \Psi^\dagger \gamma_0$

$i\bar{\Psi}\gamma_0 = i\Psi^\dagger$

conjugate momentum to Ψ is $i\Psi^\dagger$

Hamiltonian : $H = \int d^3x \ \bar{\Psi}(-i\vec{\gamma}\vec{\nabla} + m)\Psi$

spacial part only

<u>mode expansion:</u>

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^s_{\vec{p}} u^s(p) e^{-ipx} + b^{s\dagger}_{\vec{p}} v^s(p) e^{ipx} \right)$$

$$\bar{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(b^s_{\vec{p}} \bar{v}^s(p) e^{-ipx} + a^{s\dagger}_{\vec{p}} \bar{u}^s(p) e^{ipx} \right)$$

$$\{a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = (2\pi)^3 \delta(\vec{p}-\vec{q}) \delta^{rs}$$

not a Poisson bracket ! But an <u>anti-commutator</u> !

all other <u>anti-comm.</u> are 0

$$\boxed{\{a,b\} = ab + ba}$$

reason for $\{\cdot,\cdot\}$ instead of $[\cdot,\cdot]$ is that we dealing with <u>fermions</u>.

vacuum $|0\rangle$ annihilated by

$$a^s_{\vec{p}}|0\rangle = b^s_{\vec{p}}|0\rangle = 0$$

we can only have on particle with a given state:

$a^\dagger_{\vec{p}}|0\rangle$ but $a^\dagger_{\vec{p}} a^\dagger_{\vec{p}}|0\rangle = 0$ because

$$\{a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}}\} = 2(a^\dagger_{\vec{p}})^2 = 0$$

<u>Pauli exclusion principle</u> !

Hamiltonian $\quad H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$

Feynman propagator:

$$D_F(x-y) = \langle 0 | T\, \Psi(x) \bar{\Psi}(y) | 0 \rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$$