

11.3. Decomposition of tensor products

remember: last lecture characters with properties

$$\chi_{V_1 \otimes V_2}(\mu) = \chi_{V_1}(\mu) + \chi_{V_2}(\mu) \text{ and}$$

$$\chi_{V_1 \otimes V_2}(\mu) = \chi_{V_1}(\mu) \cdot \chi_{V_2}(\mu)$$

very useful to decompose tensor products

algorithm:

$$\chi_{V_1 \otimes V_2}(\mu) = \sum_{\lambda_1, \lambda_2} \text{mult}_{V_1}(\lambda_1) \text{mult}_{V_2}(\lambda_2) e^{(\mu, \lambda_1 + \lambda_2)}$$

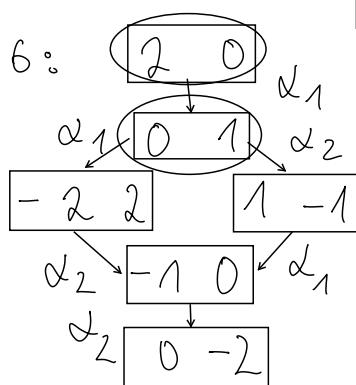
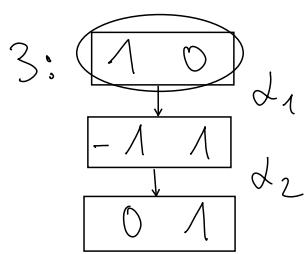
↳ lots of weights to deal with \rightsquigarrow just consider dominant weights with only non-negative λ^i 's

- 1) find the highest (with highest level) dominant weight
- 2) use it as highest weight and subtract the dominant weight system of the corresponding irrep
- 3) repeat until no weights are left

Example: $SU(3)$ $3 \otimes 3 = \bar{3} \oplus 6$

or

$$\square \otimes \square = \square \oplus \square \square$$



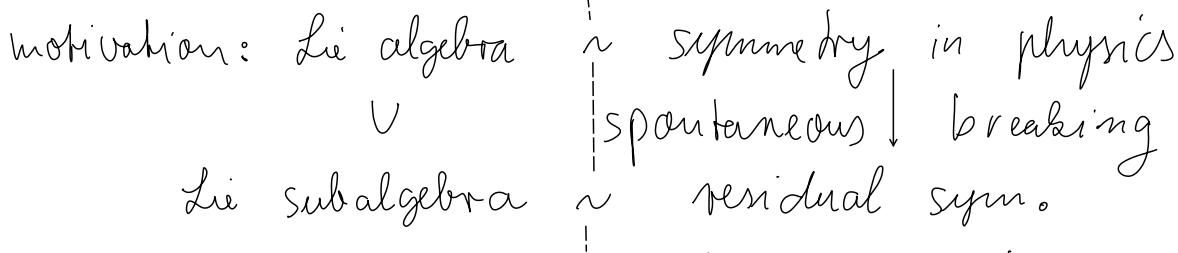
(dominant weights)

$$3 \otimes 3 \stackrel{\wedge}{=} \underbrace{\begin{array}{|c|} \hline 2 & 0 \\ \hline \end{array}}_{\text{level 2}} \oplus \underbrace{\begin{array}{|c|} \hline 0 & 1 \\ \hline \end{array}}_{\text{level 1}} \oplus 2 \begin{array}{|c|} \hline 1 & -1 \\ \hline \end{array} \oplus 2 \begin{array}{|c|} \hline -1 & 0 \\ \hline \end{array} \oplus \begin{array}{|c|} \hline 0 & -2 \\ \hline \end{array}$$

$$\rightarrow 3 \otimes 3 = \overline{3} \oplus 6$$

Lie ART: Decompose Product [Irrep $[SU(3)]^3$, Irrep $[SU(3)]^3$]

12.1. Subalgebras



We already encountered two subalgebras, the Cartan and $su(2)$.

Now: maximal subalgebras = Subalgebra h that can't be found in any larger subalg. of g other than g if self.

regular special
all step operators (roots) of other cases
 h are also step ops. of g

regular maximal subalgebras can be all found by employing extended Dynkin diagrams.

12.2 Extended Dynkin diagrams

Def: g is a simple Lie algebra with simple roots α_i and Θ denotes the highest weight of the adjoint.

The extended Dynkin diagram of g is obtained by adding a node corresponding to $-\Theta$

Example: $SU(3)$, $\Theta = \boxed{1 \ 1} = \alpha_1 + \alpha_2$

$$\begin{array}{c} \text{---} \\ | \quad | \\ -\Theta \quad \text{---} \\ | \quad | \\ \alpha_1 \quad \alpha_2 \end{array} \qquad = \boxed{2 \ -1} + \boxed{-1 \ 2}$$

→ We excluded such diagrams in the classification in section 7.2. They do not describe simple Lie algebras but are only a tool here.

g	highest root	Extended Dynkin diagram	
A_r	$1 \ 0 \dots 0 \ 1$		\hat{A}_r
B_r	$0 \ 1 \ 0 \dots 0$		\hat{B}_r
C_r	$2 \ 0 \dots 0$		\hat{C}_r
D_r	$0 \ 1 \ 0 \dots 0$		\hat{D}_r
E_6	$0 \dots 0 \ 1$		\hat{E}_6
E_7	$1 \ 0 \dots 0$		\hat{E}_7
E_8	$1 \ 0 \dots 0$		\hat{E}_8
F_4	$1 \ 0 \dots 0$		\hat{F}_4
G_2	$1 \ 0$		\hat{G}_2

By removing a node we obtain g itself or a regular (semi)simple subalgebra \mathfrak{h} .

Example: Take \hat{B}_r  and remove
 the i -th simple root:

$$-\oplus \circ \circ \circ \circ \circ \circ = D_i \oplus B_{r-i}, \quad 1 \leq i \leq r$$

or $\text{SO}(2i) \oplus \text{SO}(2r-2i+1) \subset \text{SO}(2r+1)$ or for the Lie groups
 $\text{SO}(2i) \times \text{SO}(2r-2i+1) \subset \text{SO}(2r+1)$

For \hat{A}_r we need to remove two nodes, where the second node introduces an additional $U(1)$ factor.

→ regular subalgebras of A_r are

$$A_{r-1} \times U(1) \quad \text{and} \quad A_{k-1} \times U(1) \times A_{r-k} \quad 1 < k < r$$

$$\text{or } \underset{\text{SU}(k)}{\text{SU}(r)} \times U(1) \times \underset{\text{SU}(r-k+1)}{\text{SU}(r-k+1)} \subset \text{SU}(r+1)$$

$$\underset{\text{SU}(m)}{\text{SU}(m)} \times U(1) \times \underset{\text{SU}(n)}{\text{SU}(n)} \subset \text{SU}(m+n)$$

What is the additional $U(1)$ generator?

$$t = \begin{pmatrix} i\alpha \mathbb{1}_m & 0 \\ 0 & i\beta \mathbb{1}_n \end{pmatrix} \quad t^+ = -t \quad \Rightarrow \alpha, \beta \in \mathbb{R}$$

i.e. $\alpha = \frac{m}{m+n}$ & $\beta = -\frac{n}{m+n}$

$$\text{Tr } t = 0 = i(m\alpha + n\beta)$$