# Surprisingly Complex Punctures from a Dynamical System

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in collaboration with

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# Theories of class S [Gaiotto, 2012]

6D $\mathcal{N} = (2, 0)$ SCFT

- IIB on $\mathbb{R}^{5,1} \times \mathbb{C}^2/\Gamma$, $\Gamma \subset$ SU(2) $\rightarrow$ ADE-classification [Witten, 1995]
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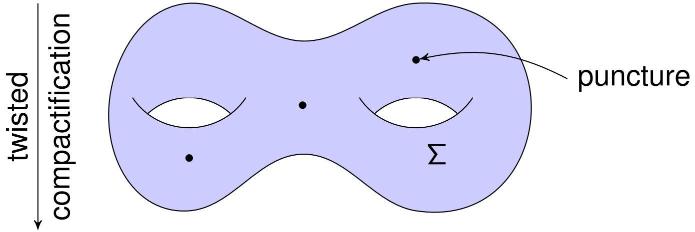


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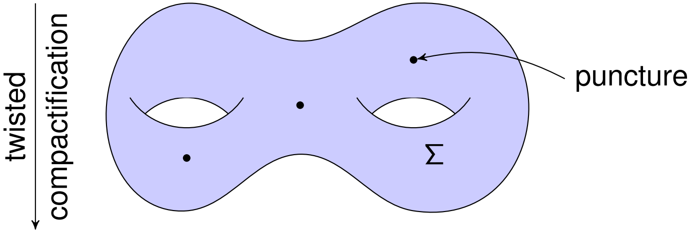
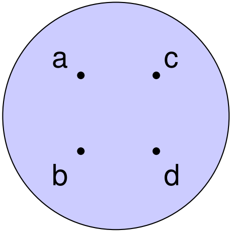



4D $\mathcal{N} = 2$ SCFT

- gauge group $G$
- flavor symmetry from punctures on $\Sigma$

# S-duality of 4D N=2 SU(2) with $N_f = 4$ flavors [Seiberg and Witten, 1994]

- flavor enhances to $\mathrm{SO(8)} \supset \mathrm{SO(2)_a} \times \mathrm{SO(2)_b} \times \mathrm{SO(2)_c} \times \mathrm{SO(2)_d}$
- exactly marginal gauge coupling $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$



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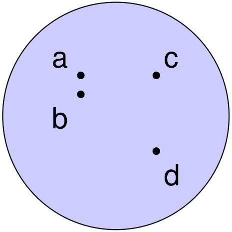


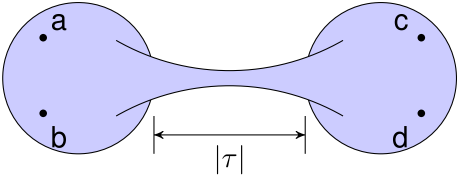

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- exactly marginal gauge coupling $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$
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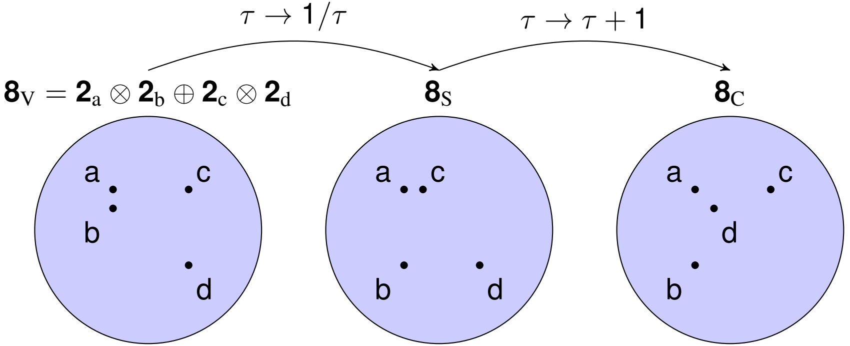
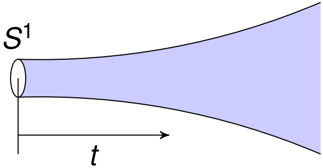
- S-duality is $\mathrm{SL}(2,\mathbb{Z})$ action on complex structure moduli space



# Constrains on punctures

- compactify 6D $\mathcal{N} = (2, 0)$ on $S^1$

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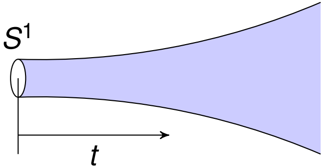


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$$\Sigma(t) = \frac{\Sigma}{t} \qquad Q(t) = \frac{Q}{t} \qquad \tilde{Q}(t) = 0$$

(in terms of $\mathcal{N}$=1 4D superfields)

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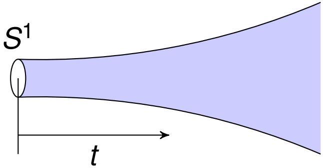


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- $\Sigma$, $Q$, $Q^\dagger$ are representations of $\mathfrak{su}(2)$

# Solutions

- $Q$ is a nilpotent $|\Gamma| \times |\Gamma|$ matrix $\to$ Jordan normal form

$$\text{e.g.} \qquad Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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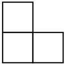
- a compact representation is the Young diagram 

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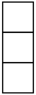
- a compact representation is the Young diagram 

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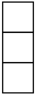
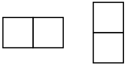
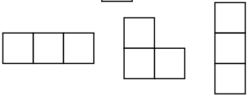
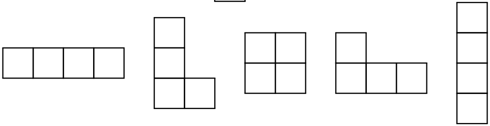
- a compact representation is the Young diagram 

| $G$ | $\Gamma$ | punctures |
|---|---|---|
| SU(2) | $A_1$ |  |
| SU(3) | $A_2$ |  |
| SU(4) | $A_3$ |  |

# ¿ Generalization to $\mathcal{N}$=1 ?

- 6D $\mathcal{N} = (1, 0)$ SCFT
- compactification $\Sigma$ with punctures $\rightarrow$ 4D $\mathcal{N}$=1 SCFTs [Razamat, Vafa, and Zafrir, 2016]

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## Challenges

- much more 6D $\mathcal{N} = (1, 0)$ than $\mathcal{N} = (2, 0)$ SCFTs [Heckman, Morrison, and Vafa, 2014]
- less constrained by SUSY

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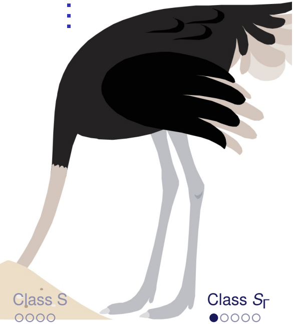

- use “simple” 6D $\mathcal{N}$=(1,0) SCFT

  $N$ M5-branes probing ADE-singularity $\mathbb{C}^2/\Gamma$
- try to classify all punctures [Heckman, Jefferson, Rudelius, and Vafa, 2017]
- harder than you might think

# Theories of Class $S_\Gamma$... [Heckman, Jefferson, Rudelius, and Vafa, 2017]

- stack of $N$ M5-branes probing ADE-singularity $\mathbb{C}^2/\Gamma$
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- stack of $N$ M5-branes probing ADE-singularity $\mathbb{C}^2/\Gamma$
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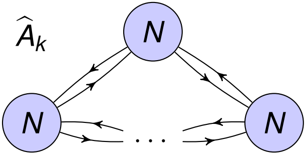
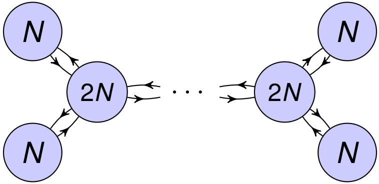
- organized according to extended Dynkin diagrams

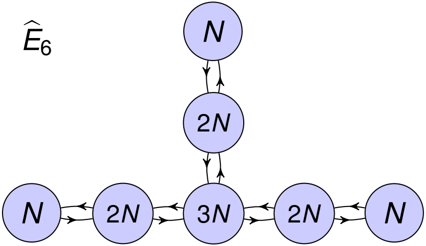
$\widehat{A}_k$




$\widehat{D}_k$




$\widehat{E}_6$




similar for $\widehat{E}_7$ and $\widehat{E}_8$

# ...and their punctures

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(in terms of $\mathcal{N}$=1 4D superfields in covering space)

- results in generalized Nahm pole equations [Heckman, Jefferson, Rudelius, and Vafa, 2

$$[\Sigma, Q] = Q \qquad [Q, \tilde{Q}] = 0$$

$$[\Sigma, \tilde{Q}] = \tilde{Q} \qquad [Q, Q^\dagger] + [\tilde{Q}, \tilde{Q}^\dagger] = \Sigma$$

plus invariance under Γ-action with

doublet $\begin{pmatrix} Q \\ \tilde{Q} \end{pmatrix}$ and singlet $\Sigma$

# A closer look at $\widehat{A}_k$ quivers

- choose $\Gamma \ni \gamma = \text{diag}(1_N, \omega 1_N, \omega^2 1_N, \ldots, \omega^k 1_N)$

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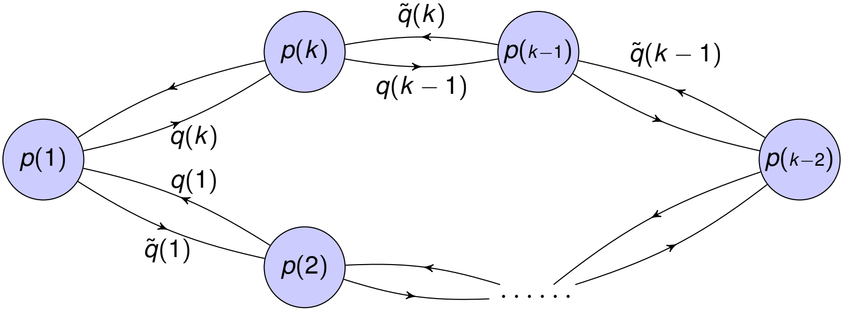
$$\Sigma = \begin{pmatrix} p(1) & & \\ & \ddots & \\ & & p(k) \end{pmatrix} \quad Q = \begin{pmatrix} & q(1) & & \\ & & \ddots & \\ & & & q(k-1) \\ q(k) & & & \end{pmatrix} \quad \tilde{Q} = \begin{pmatrix} & & & \tilde{q}(k) \\ \tilde{q}(1) & & & \\ & \ddots & & \\ & & \tilde{q}(k-1) & \end{pmatrix}$$



# Finding punctures reduces to a "simple" problem in algebra

**Problem**

1. take $N|\Gamma| \times N|\Gamma|$ matrices $Q$, $\tilde{Q}$ and $\Sigma$
2. restrict them to fit the ADE-type of $\Gamma$
3. find all fulfilling the generalized Nahm pole equations

- more complicated than we initially thought
- even for the simplest case $N$=1 $\widehat{A}_k$ quivers

# $N$=1 $\widehat{A}_k$ quivers and a dynamical system

- rewrite gen. Nahm pole eq. in terms of $q(i)$, $\tilde{q}(i)$ and $p(i)$

$$
\begin{aligned}
[Q, \tilde{Q}] = 0 \quad &\rightarrow \quad q(i+1)\tilde{q}(i+1) = q(i)\tilde{q}(i) \\
[Q, Q^\dagger] + [\tilde{Q}, \tilde{Q}^\dagger] = \Sigma \quad &\rightarrow \quad x(i) - x(i-1) = p(i) \\
[\Sigma, Q] = Q \quad &\rightarrow \quad q(i)\Big(p(i) - p(i+1)\Big) = q(i) \\
[\Sigma, \tilde{Q}] = \tilde{Q} \quad &\rightarrow \quad -\tilde{q}(i)\Big(p(i) - p(i+1)\Big) = \tilde{q}(i)
\end{aligned}
$$

with $x(i) = q(i)q(i)^* - \tilde{q}(i)\tilde{q}(i)^*$

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$$[Q,Q^\dagger] + [\tilde{Q},\tilde{Q}^\dagger] = \Sigma \quad \rightarrow \quad x(i) - x(i-1) = p(i)$$

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with $x(i) = q(i)q(i)^* - \tilde{q}(i)\tilde{q}(i)^*$

- Q is nilpotent, thus $Q^k$=$1_k \prod_{i=1}^{k} q(i)$=0 $\quad \rightarrow \quad q(i)\tilde{q}(i) = 0$
- knowing $x(i)$ is sufficient to get $q(i)$ and $\tilde{q}(i)$

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- discrete dynamical system

$$f : \begin{pmatrix} p \\ x \end{pmatrix} (i+1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix} (i) - \operatorname{sgn} x(i)$$

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- choose $x(1)$, $p(1)$ and all other $x(i)$, $p(i)$ are fixed[1]

1 In general $p(i+1)$ is unconstrained if $x(i) = 0$. We choose $p(i+1)$=$p(i)$ to formally extend the dynamical system beyond this point.

# Periodic orbits

- punctures = periodic orbits of length $k = |\Gamma|$
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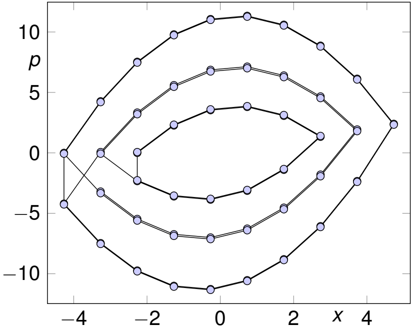


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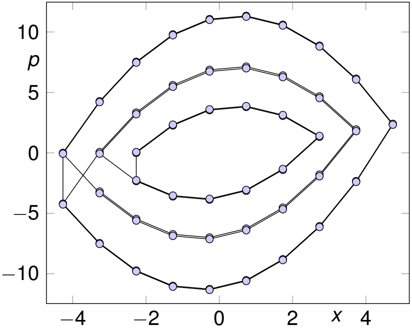


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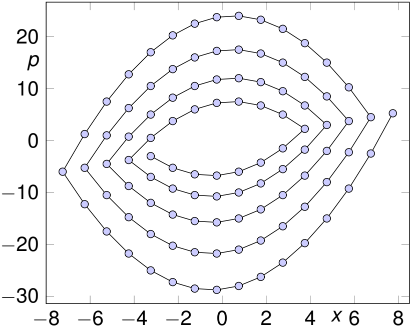

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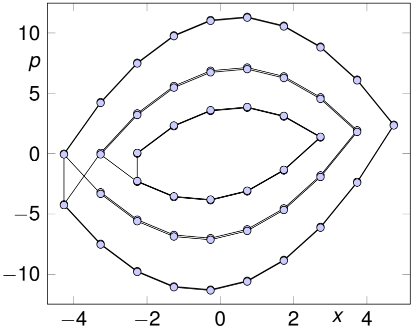


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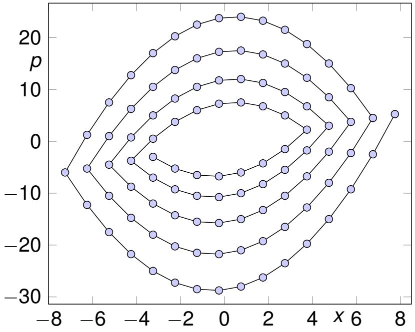
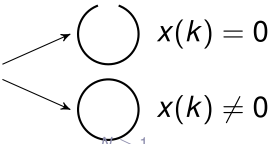
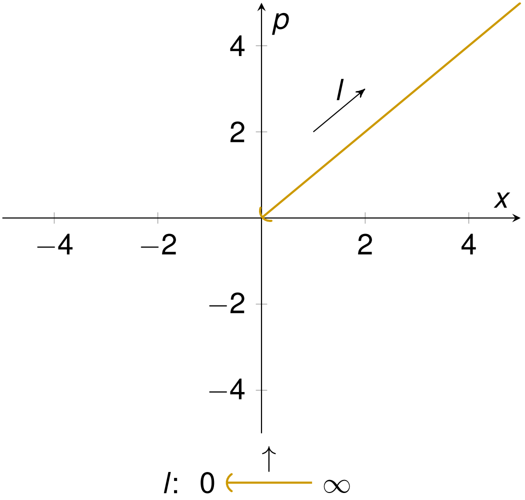



- How to find the right initial conditions?



# Divide and conquer

- if $x(k)$=0 then $p(1)$=$x(1)$=$l$, iterate line instead of point



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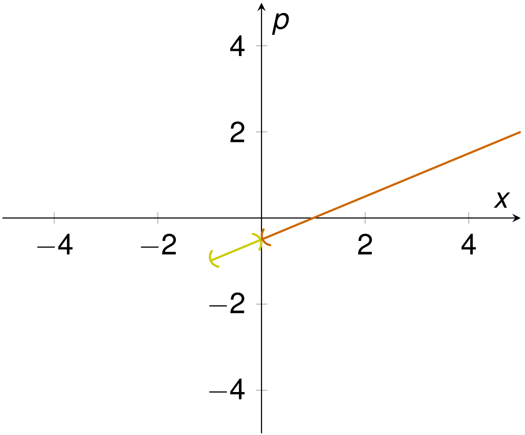


$l$: $0 \overset{\uparrow\downarrow}{\longleftrightarrow} 1/2 \overset{\uparrow\uparrow}{\longleftarrow} \infty$

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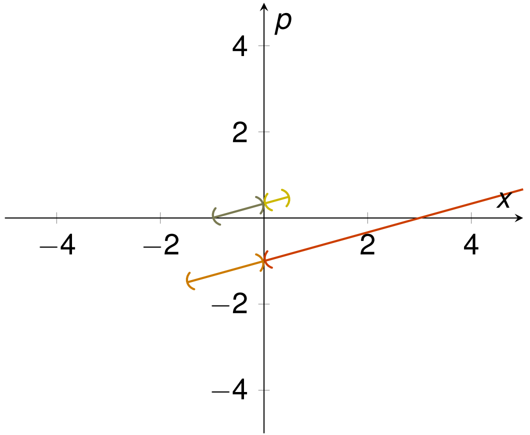


$$l: \; 0 \xleftrightarrow{\uparrow\downarrow\downarrow} 1/3 \xleftrightarrow{\uparrow\downarrow\uparrow} 1/2 \xleftrightarrow{\uparrow\uparrow\downarrow} 3/2 \xleftarrow{\uparrow\uparrow\uparrow} \infty$$

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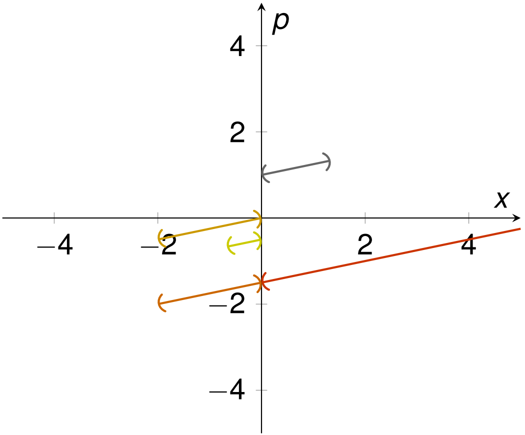


$l$: $0 \overset{\uparrow\downarrow\downarrow\uparrow}{\longleftrightarrow} 1/3 \overset{\uparrow\downarrow\uparrow\downarrow}{\longleftrightarrow} 1/2 \overset{\uparrow\uparrow\downarrow\downarrow}{\longleftrightarrow} 1 \overset{\uparrow\uparrow\uparrow\downarrow}{\longleftrightarrow} 3/2 \overset{\uparrow\uparrow\uparrow\uparrow}{\longleftarrow} \infty$

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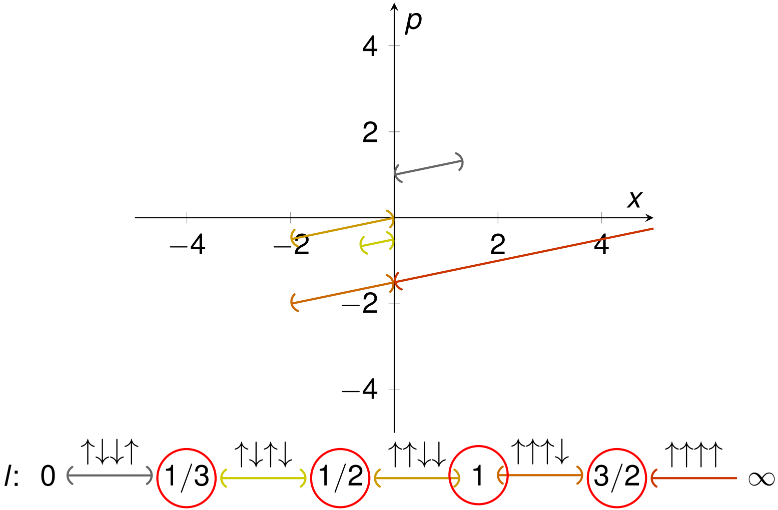
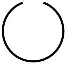
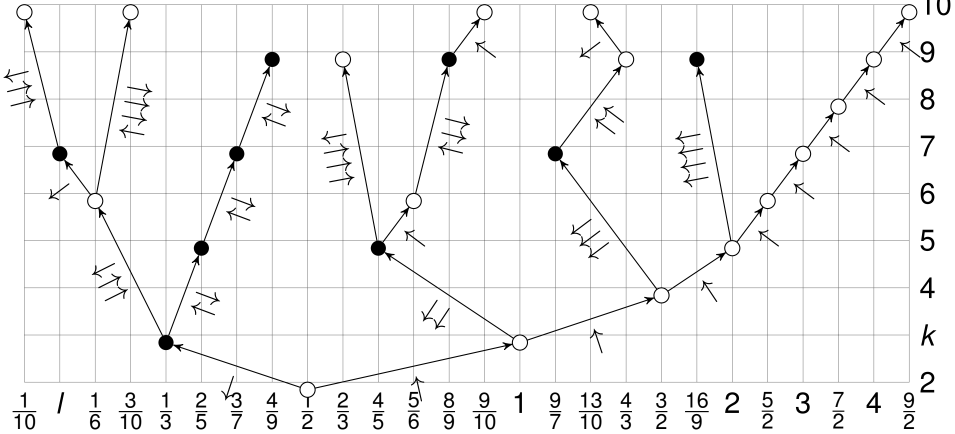


# A tree of solutions

- periodic orbits of type  $x(k) = 0$ organized in tree structure



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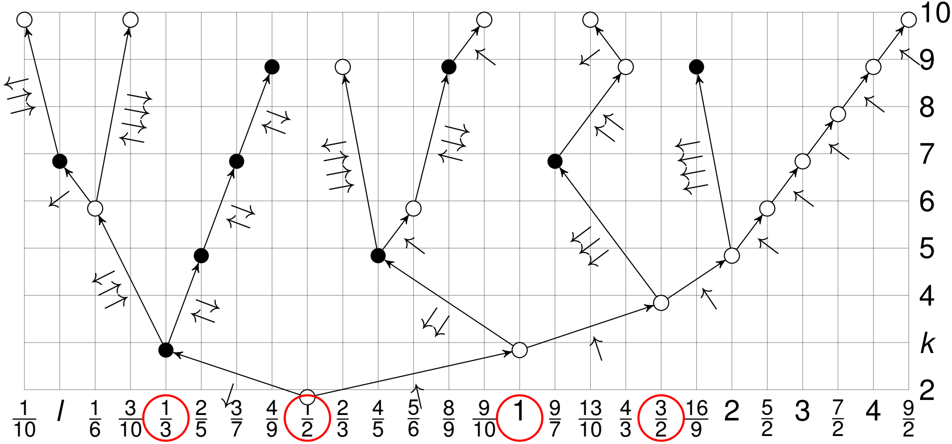

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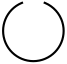
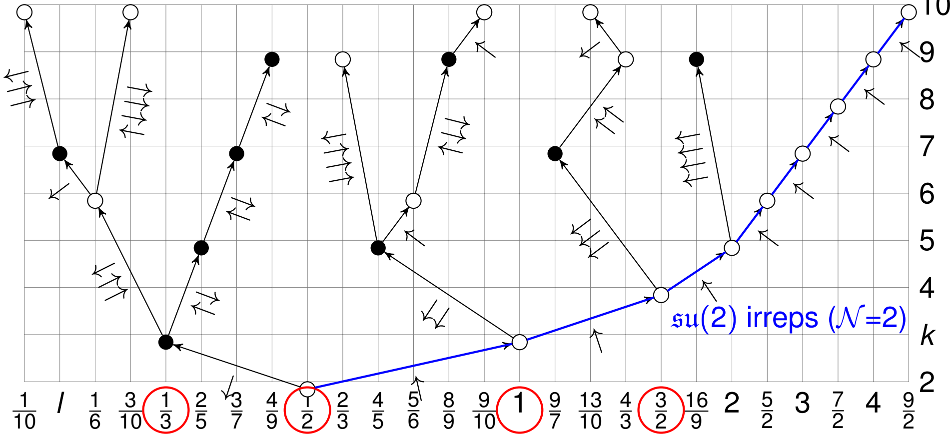
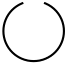
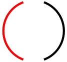
- periodic orbits of type  $x(k) = 0$ organized in tree structure



# Results, quantitative

- two kinds of solutions:

● short orbits  and ○ long orbits 

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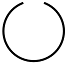
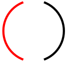
  • short orbits  and ○ long orbits 

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  $$l \in \frac{1}{2k}\Big\{ -k(k-1), -k(k-1)+4, \ldots, k(k-1) \Big\}$$

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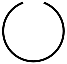
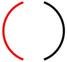
  ● short orbits  and ○ long orbits 

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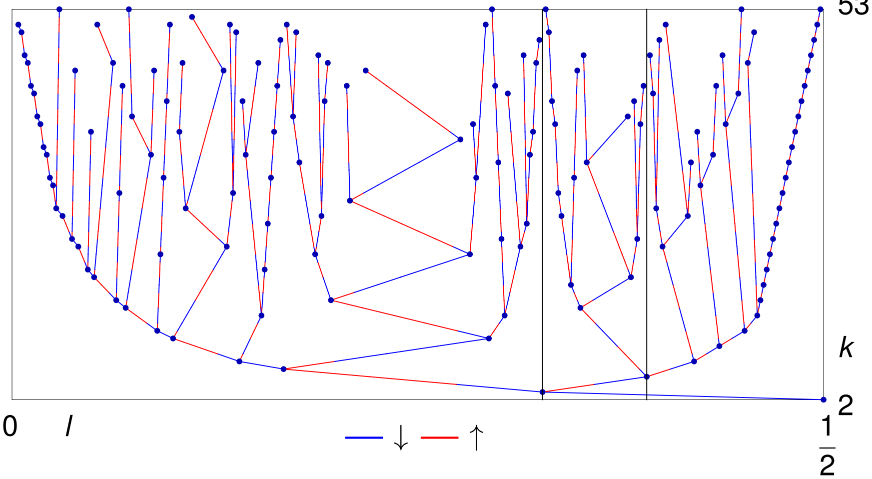
- all relevant information encoded in $\operatorname{sgn} x(i)$

- for all periodic orbits

  $$\sum_{i=1}^{k} \operatorname{sgn} x(i) = 0$$

# and qualitative

- the tree of solutions is surprisingly complex



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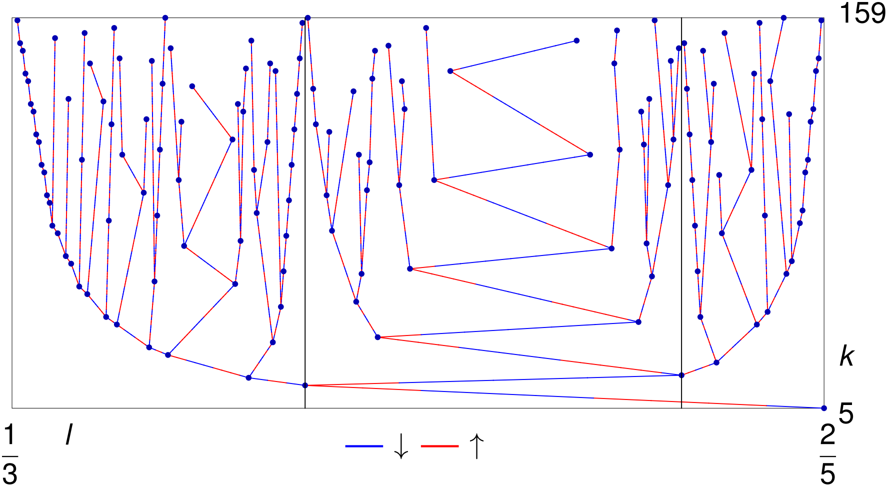

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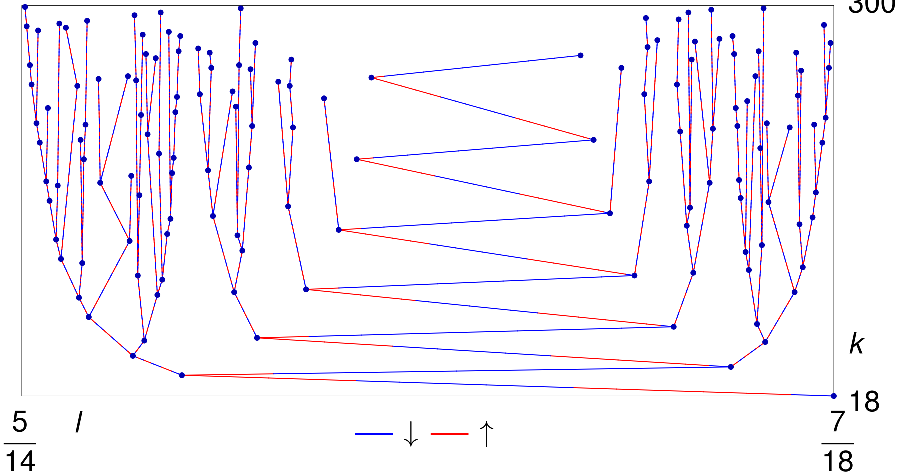

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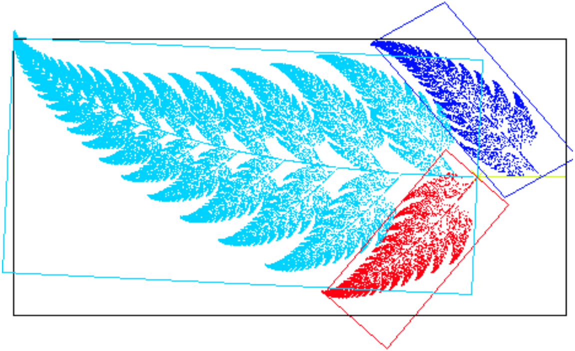

## and qualitative

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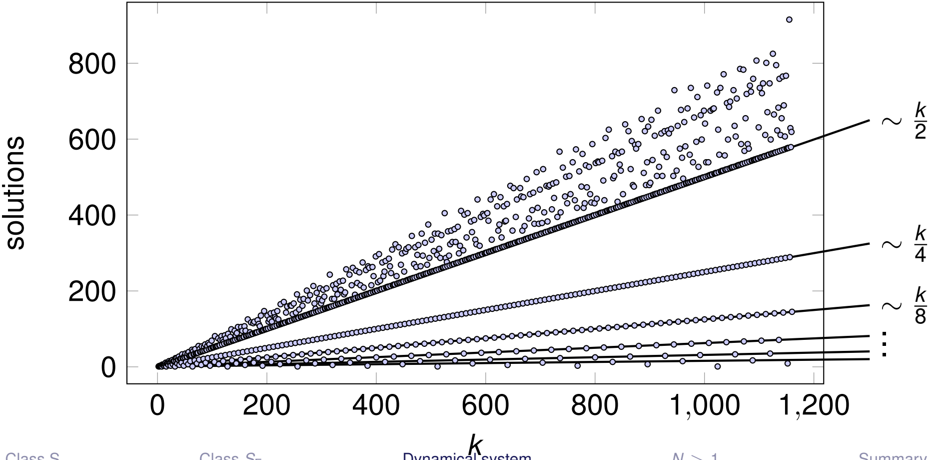
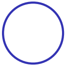
- even # of solutions has interesting structure



## Periodic orbits with ◯ $x(k) \neq 0$

- apply "divide and conquer" algorithm to

  each $p(1) \in$ set of allowed $p$ and $x(1) = I$
- slow but guaranteed to find all solutions

# Periodic orbits with  $x(k) \neq 0$

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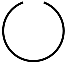
- faster: take  $x(k_1)$=0 and shift it slightly

  $p(1) = l_1$ and $x(1) = l_1 + \Delta$
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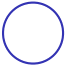
# Periodic orbits with  $x(k) \neq 0$

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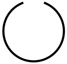
- faster: take  $x(k_1)$=0 and shift it slightly

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- iterate

  $$l_{i+1} = g(l_i) - \operatorname{sgn}\Delta$$

  until $l_n = l_1$

# Higher poles

- consider higher poles for 1/2 BPS equations

$$\Sigma(t) = \sum_{n=1} \frac{\Sigma_n}{t^{-n}} \qquad Q(t) = \sum_{n=1} \frac{Q_n}{t^{-n}} \qquad \tilde{Q}(t) = \sum_{n=1} \frac{\tilde{Q}_n}{t^{-n}}$$

- results in the puncture equations [Heckman, Jefferson, Rudelius, and Vafa, 2017]

$$\sum_{k+l=m} [\Sigma_k, Q_l] = (m-1)Q_{m-1} \qquad \sum_{k+l=m} [Q_k, \tilde{Q}_l] = 0$$

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  $m$=4: $[Q_1, \tilde{Q}_3] + [Q_2, \tilde{Q}_2] + [Q_3, \tilde{Q}_1] = 0$
- permits solution order by order with $Q_1 = Q$, $\tilde{Q}_1 = \tilde{Q}$ and $\Sigma_1 = \Sigma$

# and another dynamical system

- for $\vec{v}_m(i) = \begin{pmatrix} q_m & \tilde{q}_m & p_m \end{pmatrix}^T$

$$f_m : \ \vec{v}_m(i+1) = M_m(i)\vec{v}_m(i) + \vec{n}_m(i)$$

$$M_m(i) = \begin{cases} \begin{pmatrix} -\alpha_m(i)+\beta_m(i) & 0 & \delta_m(i) \\ 0 & \beta_m(i) & 0 \\ -\gamma_m & 0 & 1 \end{pmatrix} & \text{for } x(i) > 0 \text{ and } x(i+1) > 0 \\ \begin{pmatrix} 0 & \beta_m(i) & 0 \\ \alpha_m(i)-\beta_m(i) & 0 & -\delta_m(i) \\ -\gamma_m(i) & 0 & 1 \end{pmatrix} & \text{for } x(i) > 0 \text{ and } x(i+1) < 0 \\ \begin{pmatrix} 0 & \alpha_m(i)-\beta_m(i) & \delta_m(i) \\ \beta_m(i) & 0 & 0 \\ 0 & \gamma_m(i) & 1 \end{pmatrix} & \text{for } x(i) < 0 \text{ and } x(i+1) > 0 \\ \begin{pmatrix} \beta_m(i) & 0 & 0 \\ 0 & -\alpha_m(i)+\beta_m(i) & -\delta_m(i) \\ 0 & \gamma_m(i) & 1 \end{pmatrix} & \text{for } x(i) < 0 \text{ and } x(i+1) < 0 \end{cases}$$

$$\alpha_m(i) = \gamma_m(i)\delta_m(i)\,, \quad \beta_m(i) = \sqrt{\left|\frac{x(i)}{x(i+1)}\right|}\,, \quad \gamma_m(i) = \frac{m-2}{\sqrt{|x(i)|}}\,, \quad \delta_m(i) = \frac{m-1}{2\sqrt{|x(i+1)|}}$$

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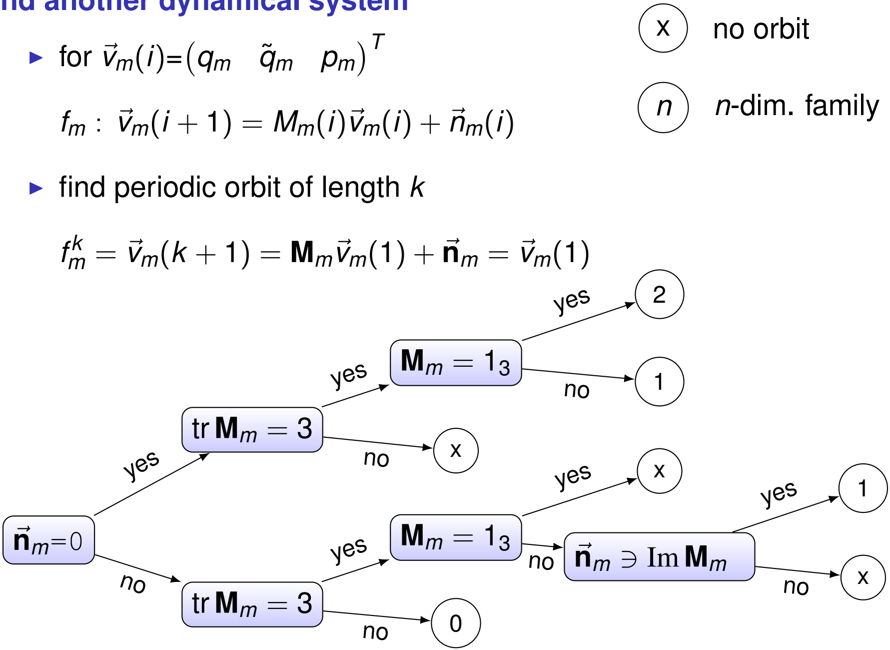

# Levels of complexity

- remember: punctures for $\mathcal{N} = 2 \rightarrow$ Young diagrams
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| | |
|---|---|
| 2 | |
| 1 | 4 |
| 0 | 3 |

decorated with $\Gamma$-charge

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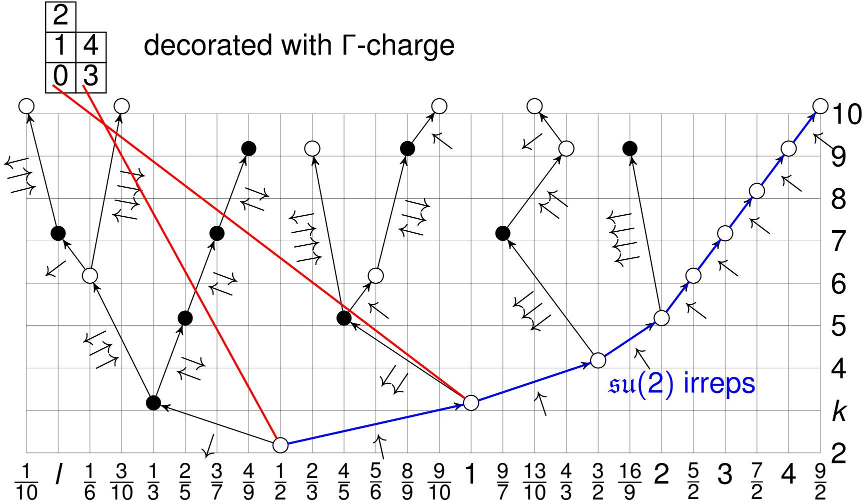

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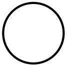
- some of them can be shifted to  with extra d.f. $x(1)$

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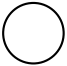
- some of them can be shifted to  with extra d.f. $x(1)$
- $x(1)$ can be tuned to gives add d.f. for h.o. punctures

## Example

$$p(1) = \frac{3}{2} \qquad x(1) = 0.8805582419579654$$

## $N > 1$ $\widehat{A}_k$ quivers are challenging

- remember $N$=1: for each "time step" solve

$$x(i) = q(i)q(i)^* - \tilde{q}(i)\tilde{q}(i)^* \quad \text{and} \quad 0 = q(i)\tilde{q}(i)$$

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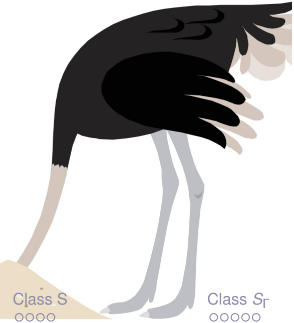
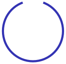


- use $N$ decoupled copy of the $N$=1 system
- embedding into $\mathfrak{u}(k)$ with irreps $>$ fundamental

# Embedding of  into $\mathfrak{su}(k)$

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  1. tensor them $n$ times
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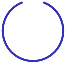
# Embedding of  into $\mathfrak{su}(k)$

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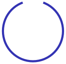
- not all irreps are allow

| k \N | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 3 | **3** | **6** | | | **15** |
| 5 | **5** | **10** | **15** | | |
| 6 | **6** | | | | |
| 7 | **7** | | **21** | | **35** |
| 9 | **9** | | | **36** | |
| 10 | **10** | | | | |

# Embedding of  into $\mathfrak{su}(k)$

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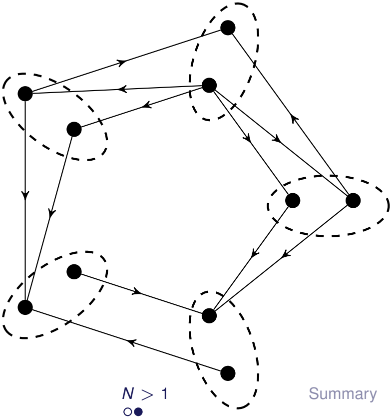

- example **10** of $\mathfrak{su}(5)$ with $l = 2/5$

## Summary

Even for the simplest class $S_\Gamma$ theories, the punctures show an amazingly rice structure compared to the $\mathcal{N} = 2$ case.

**still lots of questions**

- quantitative measure for complexity
- connection to spin chain
- statistical properties of solutions
- are the characteristic quantities for a puncture
- can we do more for $N > 1$, e.g. large $N$ limit AdS/CFT

$\vdots$